# Essay 6

*Follows Chapter 23*

# Vector Fields and Maxwell's Equations

With the introduction of the magnetic field $\vec{B}$, we now have examples of three different vector fields. We started with the velocity field $\vec{v}$ representing the flow of an incompressible fluid like water. Then we had the electric field $\vec{E}$ created by point charges like protons and electrons. And now the magnetic field which can be created by the flow of electrons in a wire.

For the electric field of point charges, we noticed that there was a mathematical analogy between the electric field and the velocity field of an incompressible fluid. Thus we could use Gauss' law and the concept of flux tubes to introduce electric field lines. Gauss' law gives us the simple rule that $Q/\varepsilon_0$ flux tubes or field lines start from a positive charge +Q or stop on a negative charge –Q. Field lines never cross or start or stop in the space between the charges. These simple rules plus an intuitive feeling for symmetry allowed us to sketch field shapes like Figure (19-15) and even solve problems like calculating the electric field of a line of charge.

For the magnetic field we used iron filings or compass needles to show us the shape of the magnetic field produced by electric currents and iron magnets. In one case, where we had a current i in a straight wire, we actually calculated the strength of the magnetic field that was going in circles about the wire. The result was

$$B = \frac{\mu_0 i}{2\pi r} \qquad (23\text{-}18)$$

We obtained this result, not from a general rule for calculating magnetic fields, but instead from Coulomb's law and our knowledge of Einstein's special theory of relativity.

There is also a general rule for calculating magnetic fields, a rule somewhat analogous to Gauss' law for electric fields. The rule is known as ***Ampere's law*** and is discussed in the Satellite Chapter 10.

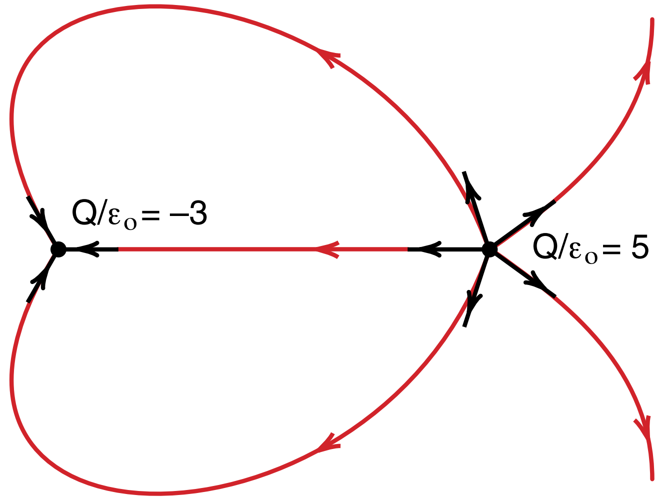


**Figure 19-15**
*Mapping electric field lines.*

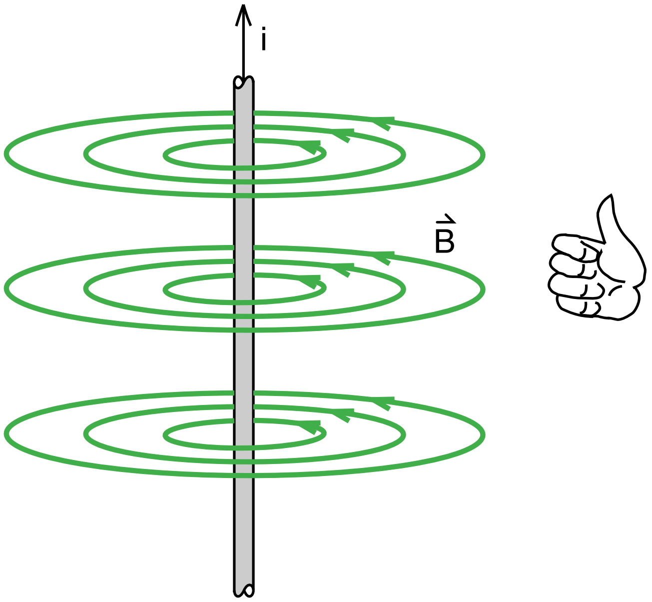


**Figure 23-14**
*Magnetic field of a straight current.*

## AMPERE'S LAW

We will give a brief example of Ampere's law as it applies to the magnetic field of a straight wire, but we will not attempt to use it here to solve problems.

To introduce Ampere's law, consider the magnetic field line $\vec{B}$ going in a circle of radius r about a current i as shown in Figure (1). Writing Equation (23-18) in the form

$$B \times 2\pi r = \mu_0 i \qquad (1)$$

we see that if we multiply the magnitude of B times the circumference of the circle $2\pi r$, the result is equal to a constant $\mu_0$ times the current i flowing through the circle.

Ampere's law is a generalization of this result. It tells you to walk around a complete path, coming back to the point where you started. The circle in Figure (1) is such a path. With each step you take while walking, multiply the length of your step by the strength of the magnetic field in the direction of your step, and record the result. When you get back to your starting point and add up all your results, the sum will be the constant $\mu_0$ times the total amount of current i flowing through your path. In the case of Figure (1), $\vec{B}$ is always pointing in the direction of the circular path, so that all we have to do is multiply B times the total length of the path $2\pi r$, to get $\mu_0 i$.

In Satellite Chapter 10 ***Vector fields and Ampere's Law***, we do a more detailed introduction to Ampere's law, and then use the law for calculating other magnetic field structures, like the field inside a coil of wire.

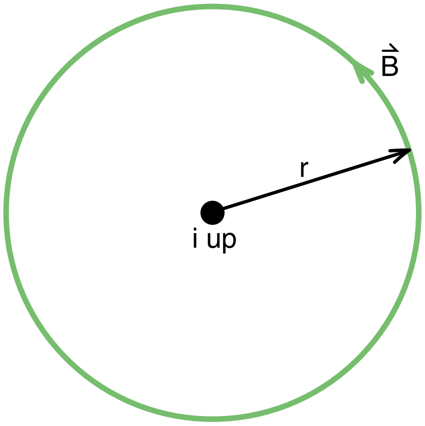


**Figure 1**
*Magnetic field of a current.*

## TWO KINDS OF VECTOR FIELDS

The electric field $\vec{E}$ of a point charge, and the magnetic field $\vec{B}$ of a straight current, compared in Figure (2), are examples of two different kinds of vector fields. The electric field of a point charge is a clear example of what we call a ***diverging*** kind of vector field. The field lines diverge, or come straight out of the point charge.

The magnetic field surrounding a wire is an example of what we call a ***circulating*** vector field. Here, instead of diverging out of its source, it circulates around its source. One of the accomplishments of the mathematics of vector fields is to show that these two kinds of fields have completely distinct sources. For example, electric charges at rest produce only diverging electric fields, and electric currents produce only circulating magnetic fields.

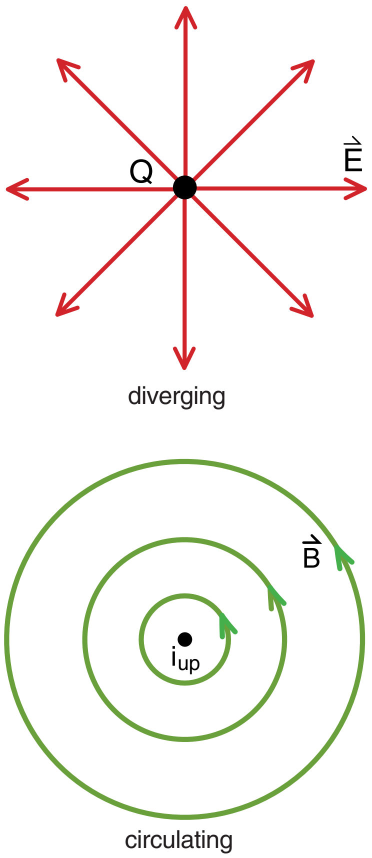


**Figure 2**
*Two kinds of vector fields.*

As a test of this idea, we can ask what kind of a vector field is the velocity field $\vec{v}$ of an incompressible fluid like water? Near the beginning of Chapter 18 on fluids we discussed the velocity field of a point source of water, seen in Figure (18-6). In that figure we imagined that water molecules were being created inside a small magic sphere. Beyond that sphere there was no more magic, and we used the continuity equation to show that the velocity field dropped off as $1/r^2$ as the fluid flowed out through larger spheres whose area increased as $r^2$ .

The problem with that example is that for an incompressible fluid, ***there are no magic spheres***. We do not see water actually diverging outward as shown in Figure (18-6). Instead we see flow patterns like the bathtub vortex of Figures (18-25) and (18-26) or Hurricane Allen in Figure (18-27). Because there are no magic spheres creating water molecules, the only kind of velocity field we see in an incompressible fluid is a circulating velocity field $\vec{v}$. You will not see the diverging velocity field of Figure (18-6).

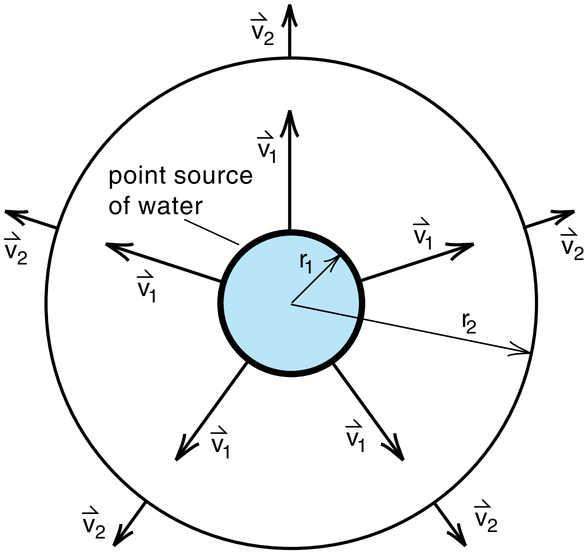


**Figure 18-6**
***Water molecules are created inside the small magic sphere. This creates a diverging velocity field out from the small sphere.***

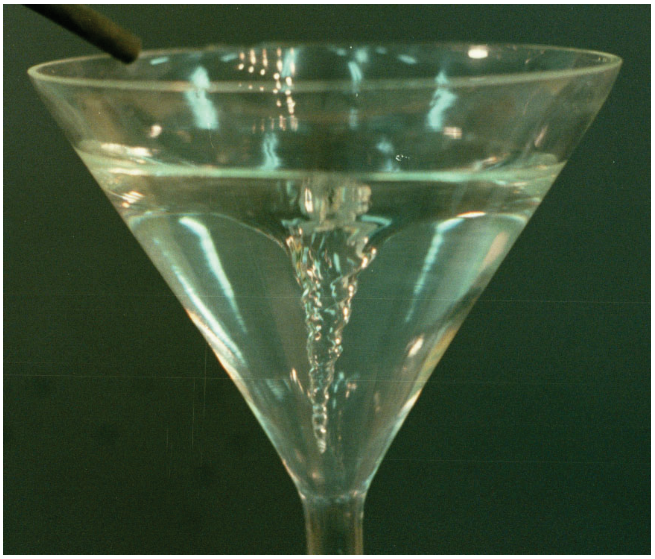


**Figure 18-25**
***Bathtub vortex in a funnel. We stirred the water before letting it drain out.***

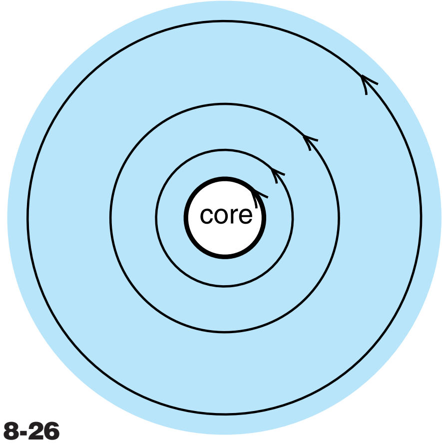


**Figure 18-26**
***Velocity field of the bathtub vortex.***

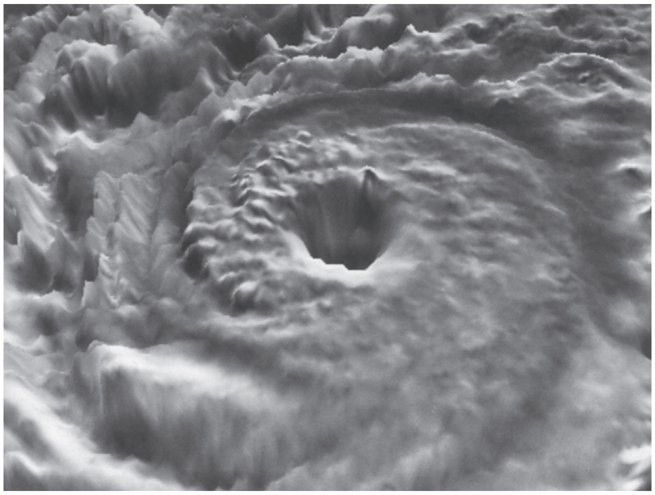
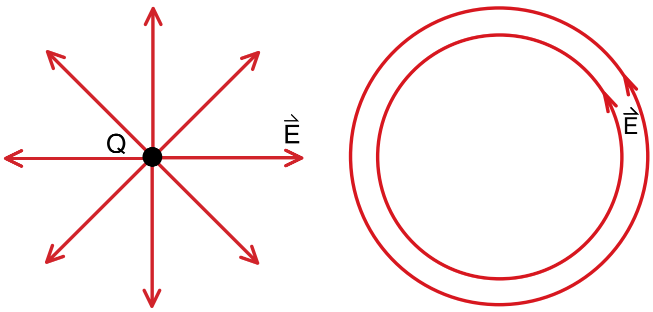


**Figure 18-27**
***Eye of hurricane Allen viewed from a satellite.***

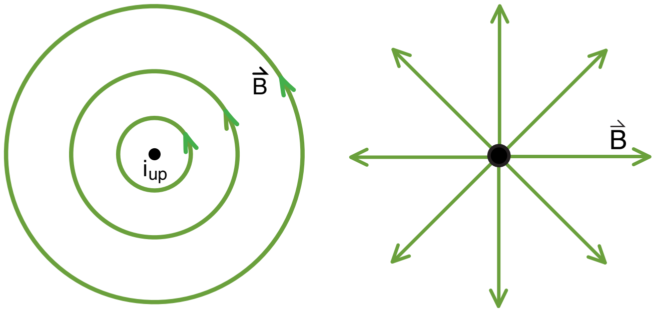
## ELECTROMAGNETIC FIELDS

We will now focus our attention on the two vector fields involved in electricity and magnetism, $\vec{E}$ and $\vec{B}$. Because each of these fields could, at least in principle, have two components, a diverging kind and a circulating kind, there are four possible electric and magnetic field types indicated in Figure (3). There is the diverging electric and circulating magnetic fields we saw in Figures (3a) and (3b). The other two possibilities are a circulating electric and a diverging magnetic field drawn in Figures (3c) and (3d).

It turns out that each kind of field has its own special kind of source and its own special law for calculating the field produced by that source. For the diverging electric field, the source is electric charge, and the law determining the field produced is Gauss' law. For the circulating magnetic field, the source is electric current and the law used to calculate the resulting magnetic field is Ampere's law.



3a) diverging electric field 3c) circulating electric field



3b) circulating magnetic field 3d) diverging magnetic field

**Figure 3**
*Four possible kinds of electric and magnetic fields.*

For the diverging magnetic field shown in (3d), the source is the so called ***magnetic monopole***. Some theories of the early universe predict that magnetic monopoles should have been created shortly after the big bang. Physicists have spent years looking for a magnetic monopole, but so far have found none. Until they do find one, we have a very simple rule rule for calculating the diverging kind of magnetic field—***there is none***!

It may be a surprise, but the circulating kind of electric field not only exists, it plays a very important role in modern life. ***The source of a circulating electric field is a changing magnetic field***. The practical applications which we could not live without is the ***electric generator***. By changing the magnetic field passing through a wire loop you create an electric voltage around the loop, a voltage that can be used by a power company to provide electric voltage to your house.

The law we use to calculate the circulating kind of electric field is ***Faraday's law*** discussed in the Satellite Chapter 11. Of all the satellite chapters, this may be the one that should be included as part of the regular course, if time permits. In that chapter we discuss and carry out an important experiment that helped lead Einstein to the special theory of relativity.

Satellite Chapter 10
***Vector Fields***
***and Ampere's Law***

Satellite Chapter 11
***Faraday's Law***
***AND Generators***

Satellite Chapter 12
***Magnetic Force***
***on an Electric Current***

***Related satellite chapters.***

## MAXWELL'S EQUATIONS

The four equations governing the behavior of the four components of electric and magnetic fields are known as ***Maxwell's equations***. They are ***Gauss' law*** for diverging electric fields, ***Ampere's law*** for circulating magnetic fields, ***Faraday's law*** for circulating electric fields, and the ***absence of magnetic monopoles*** for the absence of divergent magnetic fields. Then why, with Gauss, Ampere and Faraday all preceeding Maxwell, do we call these Maxwell's equations?

The answer begins with the fact that Maxwell discovered another source for the magnetic field. We mentioned that a changing magnetic field could create a circulating electric field. Maxwell discovered that a changing electric field would act as a source of a circulating magnetic field.

At this point Maxwell saw a new possibility. Out in empty space where there are no charges or currents, he could still create electric and magnetic fields. He had an equation for how a changing magnetic field could create an electric field. The electric field being created this way could in turn create a magnetic field. The magnetic field being created this way could create an electric field . . . etc. In other words, electric and magnetic fields could feed off of each other and travel in a wavelike manner through empty space. He calculated the speed of this ***electromagnetic*** wave and the result was the speed c given by

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \frac{\text{meters}}{\text{sec}}$$

which is the speed of light.

As a result, Maxwell saw that he had discovered what a light wave was—a wave of electric and magnetic fields feeding off of each other, traveling as a wave through empty space. In the next chapter we study the structure of this electromagnetic wave.