# Chapter 10
# Energy

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*In principle, Newton's laws relating force and acceleration can be used to solve any problem in mechanics involving particles whose size ranges from that of specks of dust to that of planets. In practice, many mechanics problems are too difficult to solve if we try to follow all the details and analyze all the forces involved. For instance* $\vec{f} = m\vec{a}$ *presumably applies to the motion of the objects involved in the collision of two automobiles, but it would be an enormous task to study the details of the collision by analyzing all the forces involved.*

*In a complicated problem, we cannot follow the motion of all the individual particles; instead we look for general principles that follow from Newton's laws and apply these principles to the system of particles as a whole. We have already discussed two such general principles: the laws of conservation of linear and angular momentum. We have found that if two cars traveling on frictionless ice collide and stick together, we can use the law of conservation of linear momentum to calculate their resulting motion. We do not have to know how they hit or any other details of the collision.*

*In our discussion of satellite motion, we saw that there was another quantity, which we called energy, that was conserved. Our formula for the total energy of the satellite was* $E_{total} = 1/2\ mv^2 - Gmm_e/r$ *where* $1/2\ mv^2$ *was called the* ***kinetic energy*** *and* $-Gmm_e/r$ *the* ***gravitational potential energy*** *of the satellite. We saw that* $E_{total}$ *did not change its value as the satellite went around its orbit.*

*It turns out that energy is a much more complex subject than we might suspect from the discussion of satellite motion. There are many forms of energy, such as electrical energy, heat energy, light energy, nuclear energy and various forms of potential energy. Sometimes there is a simple formula for a particular form of energy, but sometime it may be hard even to figure out where the energy has gone. Despite the complexity, one simple fact remains, if we look hard enough we find that energy is conserved.*

*If, in fact, it were not for the conservation of energy, we would not have invented the concept in the first place. Energy is a useful concept only because it is conserved.*

*What we are going to do in this chapter is first take a more general look at the idea of a conservation law, and then see how we can use energy conservation to develop formulas for the various forms of energy we encounter. We will see, for example, where the formula* $1/2\ mv^2$ *for kinetic energy comes from, and we will show how the formula* $-Gmm_e/r$ *for gravitational potential energy reduces to a much simpler formula when applied to objects falling near the surface of the earth.*

## CONSERVATION OF ENERGY

Because energy comes in different forms, it is more difficult to state how to compute energy than how to compute linear momentum. But, as we shall see, it is not necessary to state all the formulas for all the different forms of energy. If we know the formula for some forms of energy, we can use the law of conservation of energy to deduce the other formulas as we need them. How a conservation law can be used in this way is illustrated in the following story, told by Richard Feynman in *The Feynman Lectures on Physics* (Vol. I, Addison-Wesley, Reading, Mass., 1963).

*"Imagine a child, perhaps 'Dennis the Menace,' who has blocks that are absolutely indestructible, and cannot be divided into pieces. Each is the same as the other. Let us suppose that he has 28 blocks. His mother puts him with his 28 blocks into a room at the beginning of the day. At the end of the day, being curious, she counts the blocks very carefully, and discovers a phenomenal law—no matter what he does with the blocks, there are always 28 remaining! This continues for a number of days, until one day there are only 27 blocks, but a little investigating shows there is one under the rug—she must look everywhere to be sure that the number of blocks has not changed.*

*One day, however, the number appears to change—there are only 26 blocks. Careful investigation indicates that the window was open, and upon looking outside, the other two blocks are found. Another day careful count indicates that there are 30 blocks! This causes considerable consternation, until it is realized that Bruce came to visit, bringing his blocks with him, and he left a few at Dennis' house. After she had disposed of the extra blocks, she closes the window, does not let Bruce in, and then everything is going along all right, until one time she counts and finds only 25 blocks. However, there is a box in the room, a toy box, and the mother goes to open the toy box, but the boy says, 'No, do not open my toy box,' and screams. Mother is not allowed to open the toy box. Being extremely curious, and somewhat ingenious, she invents a scheme! She knows that a block weighs 3 ounces, so she weighs the box at a time when she sees 28 blocks, and it weighs 16 ounces. The next time she wishes to check, she weighs the box again, subtracts 16 ounces and divides by 3. She discovers the following:*

$$\begin{pmatrix} \textit{number of} \\ \textit{blocks seen} \end{pmatrix} + \frac{\textit{weight of box} - 16\ oz}{3\ oz} = \textit{constant}$$

*There then appear to be some gradual deviations, but careful study indicates that the dirty water in the bathtub is changing its level. The child is throwing blocks into the water, and she cannot see them because it is so dirty, but she can find out how many blocks are in the water by adding another term to her formula. Since the original height of the water was 6 inches and each block raises the water a quarter of an inch, this new formula would be*

$$\begin{pmatrix} \textit{number of} \\ \textit{blocks seen} \end{pmatrix} + \frac{\textit{weight of box} - 16\ oz}{3\ oz} + \frac{\textit{height of water} - 6\ \textit{inches}}{1/4\ \textit{inch}} = \textit{constant} \tag{1}$$

*In the gradual increase in the complexity of her world, she finds a whole series of terms representing ways of calculating how many blocks are in places where she is not allowed to look. As a result of this, she finds a complex formula, a quantity which* ***has to be computed****, which always stays the same in her situation."*

Similarly, we will find a series of terms representing ways of calculating various forms of energy. Unlike the story, where some blocks are actually seen, we cannot see energy; all of the terms in our equation for energy must be computed. But if we have included enough terms and have not neglected any forms of energy, the numerical value of all the terms taken together will not change; that is, we will find that energy is conserved.

It is not necessary, however, to start with the complete energy equation. We will begin with one term. Then, as the complexity of our world increases, we will add more terms to the equation so that energy remains conserved.

## MASS ENERGY

On earth, the greatest supply of useful energy ultimately comes from the sun, mainly as sunlight, which is a form of radiant energy. The energy we obtain from fossil fuel, such>é” coal and wood, and the energy we get from hydroelectric dams came originally from the sun. On a clear day, the sun delivers as much energy to half a square mile of tropical land as was released by the first atomic bomb. In about 1 millionth of a second, the sun radiates out into space an amount of energy equal to that used by all of mankind during an entire year.

The sun emits radiant energy at such an enormous rate that if it burned like a huge lump of coal, it would last about 5000 years before burning out. Yet the sun has been burning at nearly its present rate for over 5 billion years and should continue burning for another 5 billion years. How the sun could emit all of this energy was explained in 1905 when Einstein discovered that mass and energy are related through the well-known equation

$$\boxed{E = mc^2} \tag{2}$$

where **E** is energy, **m** mass, and **c** the speed of light.

The sun's source of energy is the tiny fraction of its mass that is being converted continually to radiant energy through nuclear reactions. Similar processes occur when the hydrogen bomb is exploded. To indicate the amount of energy that is in principle available as mass energy, imagine that the mass of a 5–cent piece (5 gm) could be converted entirely into electrical energy. This electrical energy would be worth several million dollars. The problem is that we do not have the means available to convert mass completely into a useful form of energy. Even in the nuclear reactions in the sun or in the atomic or hydrogen bombs, only a few tenths of 1% of the mass is converted to energy.

Since most of the energy in the universe is in the form of mass energy, we shall begin to develop our equation for energy with Einstein's formula $E = mc^2$. As we mentioned, we will add terms to this equation as we discover formulas for other forms of energy.

## Ergs and Joules

Our first step will be to use the Einstein energy formula to obtain the dimensions of energy. In the CGS system of units we have

$$E = m\ \text{gm} \times c^2\ \frac{\text{cm}^2}{\text{sec}^2} = mc^2\ \frac{\text{gm cm}^2}{\text{sec}^2}$$

The set of dimensions gm $\text{cm}^2/\text{sec}^2$ is called an ***erg***.

$$1\ \frac{\text{gm cm}^2}{\text{sec}^2} = 1\ \text{erg} \qquad \left(\textbf{\textit{CGS units}}\right)$$

In the MKS system of units, we have

$$E = m\ \text{kg} \times c^2\ \frac{\text{m}^2}{\text{sec}^2} = mc^2\ \text{kg}\ \frac{\text{m}^2}{\text{sec}^2}$$

where the set of dimensions of kg $\text{m}^2/\text{sec}^2$ is called a ***joule***.

$$1\ \frac{\text{kg m}^2}{\text{sec}^2} = 1\ \text{joule} \qquad \left(\textbf{\textit{MKS units}}\right)$$

It turns out that for many applications the MKS joule is a far more convenient unit of energy than the CGS erg. A 100-watt light bulb uses 100 joules of energy per second, or 1 billion ergs of energy per second. The erg is too small a unit of energy for many applications, and it is primarily for this reason that the MKS system of units is more often used than the CGS system. This is particularly true when dealing with electrical phenomena.

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**Exercise 1**

(a) Use dimensions to determine how many ergs there are in a joule. (Check your answer against the statement that a 100-watt bulb uses 100 joules or $10^9$ ergs of energy per second.)

(b) As you may have guessed, a 1 watt light bulb uses 1 joule of energy per second. How many joules of energy does a 1000 watt bulb or heater use in one hour. (This amount of energy is called a ***kilowatt hour*** *(abbreviated* **kwh**) and costs a home owner about 10 cents when supplied by the local power company.)

(c) If a 5-cent piece (which has a mass of 5 grams) could be converted entirely to energy, how many kilowatt hours of energy would it produce? What would be the value of this energy at a rate of 10¢ per kilowatt hour?

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## KINETIC ENERGY

From the recoil definition of mass (Chapter 6), we saw that the mass of an object increases with speed, becoming very large when the speed of the object approaches the speed of light. The formula for the increase in mass with speed was simply

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \qquad (6\text{-}14)$$

where $m_0$ is the mass of the particle at rest (the ***rest mass***). When we combine this formula with Einstein's equation $E = mc^2$, we get as the equation for the energy of a moving particle

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} \qquad (3)$$

According to Equation (3), when a particle is at rest (v = 0), its energy is given by

$$E_0 = m_0 c^2 \quad \textit{rest energy} \qquad (4)$$

This energy $m_0c^2$ is called the ***rest energy*** of the particle. As a particle begins to move, its mass, and therefore its energy, increases. The *extra* energy that a particle acquires as a result of its motion is called ***kinetic energy***. If $mc^2$ is the total energy, then the formula for the particle's kinetic energy is

$$\text{kinetic energy} = \text{total energy} - \text{rest energy}$$

$$KE = mc^2 - m_0 c^2 \qquad (5)$$

### Example 1

The muons in the motion picture *Time Dilation of the μ–Meson (Muon) Lifetime* moved at a speed of .995c. By what factor did their mass increase and what is their kinetic energy?

***Solution***: The first step is to calculate $\sqrt{1 - v^2/c^2}$ for the muons. An easy way to do this is as follows:

$$v = .995\,c$$

$$\frac{v}{c} = .995 = 1 - .005$$

$$\frac{v^2}{c^2} = (1 - .005)^2$$

$$= 1 - 2(.005) + (.005)^2$$

$$= 1 - .01 + \cancel{.000025}$$

We have neglected .000025 compared to .01 because it is so much smaller. We now have

$$1 - \frac{v^2}{c^2} \approx 1 - (1 - .01) = .01$$

$$\sqrt{1 - \frac{v^2}{c^2}} \approx \sqrt{.01} = .1 = \frac{1}{10}$$

(This procedure is discussed in more detail in the section on approximation formulas in Chapter 1.)

Now that we have $\sqrt{1 - v^2/c^2} = 1/10$ for these muons, we can calculate their relativistic mass

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} = \frac{m_0}{1/10} = 10\,m_0$$

Thus the mass of the muons has increased by a factor of 10. The total energy of the muons is

$$E = mc^2 = (10m_0)c^2 = 10(m_0c^2)$$

Hence, their total energy is also 10 times their rest energy. Their increase in energy, or their kinetic energy, is

$$KE = mc^2 - m_0c^2 = 10m_0c^2 - m_0c^2$$

$$KE = 9\,m_0c^2$$

This kinetic energy $9m_0c^2$ is the amount of ***additional energy*** that is required to get muons moving at a speed $v = 0.995c$.

**Exercise 2**

Assume that an electron is traveling at a speed v = .99995c.

(a) What is $\sqrt{1-v^2/c^2}$ for this electron?

(b) By what factor has its mass increased over its rest mass?

(c) By what factor has its total energy increased over its rest energy?

(d) The rest mass of an electron is $m_0 = 0.911 \times 10^{-27}$ gm. What is its rest energy (in ergs)?

(e) What is the total energy (in ergs) of this electron?

(f) What is the kinetic energy of this electron in ergs?

## Slowly Moving Particles

In Example 1, where the particle (muon) was moving at nearly the speed of light, we determined its increase in mass and its kinetic energy by calculating $\sqrt{1-v^2/c^2}$. However, when a particle is moving much slower than the speed of light, for instance, 1000 mi/sec or less, there is an easier way to calculate the energy of the object than by evaluating $\sqrt{1-v^2/c^2}$ directly.

In the section on approximation formulas in Chapter 1, it was shown that when v/c is much less than 1, then we can use the approximate formula

$$\frac{1}{\sqrt{1-\alpha}} \approx 1 + \frac{\alpha}{2} \qquad (1\text{-}25)$$

to get

$$\frac{1}{\sqrt{1-v^2/c^2}} \approx 1 + \frac{v^2}{2c^2} \qquad (6)$$

The approximate formula $(1 + v^2/2c^2)$ is much easier to use than $1/\sqrt{1-v^2/c^2}$. Moreover, if v/c is a small number, then the formula is quite accurate, as illustrated in Table 1. It should be noted however that when v becomes larger than about .1c, the approximation becomes less accurate. When we reach v = c, the exact formula is $1/\sqrt{1-v^2/c^2} = \infty$ but the approximate formula gives $1 + v^2/2c^2 = 1.5$. At this point the approximate formula is no good at all!

Table 1 Numerical check of the Approximation Formula $\frac{1}{\sqrt{1-v^2/c^2}} \approx 1 + \frac{v^2}{2c^2}$

| v | *value of exact formula* $\frac{1}{\sqrt{1-v^2/c^2}}$ | *value of approximate formula* $1 + \frac{v^2}{2c^2}$ |
|---|---|---|
| .01c | 1.000050003 | 1.000050000 |
| .1c | 1.005037 | 1.005000 |
| .2c | 1.0206 | 1.0200 |
| .3c | 1.048 | 1.045 |
| .5c | 1.148 | 1.125 |
| .7c | 1.41 | 1.25 |
| .9c | 2.30 | 1.40 |
| .99c | 7.1 | 1.49 |
| c | ∞ | 1.5 |

If we use Equation (6), the total energy of a particle becomes

$$E = mc^2$$

$$= m_0c^2\left(\frac{1}{\sqrt{1 - v^2/c^2}}\right) \quad \textit{exact formula}$$

$$\approx m_0c^2\left(1 + \frac{v^2}{2c^2}\right) \quad \textit{approximate formula}$$

$$\approx m_0c^2 + m_0\cancel{c^2}\frac{v^2}{2\cancel{c^2}}$$

The factor $c^2$ cancels in the second term, and we are left with the approximate formula

$$E \approx m_0c^2 + \frac{1}{2}m_0v^2 \quad \begin{array}{l}\textit{approximate formula}\\ \textit{for particles moving}\\ \textit{at speeds less than}\\ \textit{about .1c}\end{array} \tag{7}$$

Since Equation (7) contains the approximation made in Equation (6), it is not valid for particles traveling faster than about one tenth of the speed of light. For particles traveling at nearly the speed of light, we must use $E = m_0c^2/\sqrt{1 - v^2/c^2}$ . But for particles traveling as slowly as a few thousand miles an hour or less, Equation (6) is so accurate that any error would be difficult to detect.

For all but the last section of this chapter, we will confine our discussion to the energy of objects traveling at slow speeds, where Equation (7) is not only accurate, but is the simplest equation to use. When we look at this equation, we can see that the mass energy $E = mc^2$ is now written in two distinct parts $m_0c^2$, which is the rest mass energy, and $1/2m_0v^2$, which is the energy of motion or kinetic energy

$$\boxed{E = m_0c^2 \left(\begin{array}{c}\textit{rest}\\ \textit{energy}\end{array}\right) + \frac{1}{2}m_0v^2 \left(\begin{array}{c}\textit{kinetic}\\ \textit{energy}\end{array}\right)} \tag{7a}$$

Written in this way, our equation for total energy is beginning to resemble Equation (1), which was used to determine the number of blocks in Dennis' room. We now have two terms representing two different kinds of energy.

It is worth noting that, at one time, only the kinetic energy term $1/2m_0v^2$ in Equation 7 was recognized as a form of energy. Before 1905, it was not known that $m_0c^2$ should be included in the equation for conservation of energy, because no one had ever observed the rest mass of an object to change. The first evidence that the rest energy had to be included came from the study of nuclear reactions. In these reactions enormous amounts of energy were released, producing a detectable change in the nuclear rest masses.

So long as an object is moving at a speed of .1c or less, the kinetic energy of that object will be far less than its rest mass energy. For example, let us compare the kinetic energy to the rest mass energy of a 10–gm pistol bullet that travels with a speed of about 300 m/sec. Using MKS units, we find that the bullet's kinetic energy (KE) is

$$KE = \frac{1}{2}m_0v^2$$

$$= \frac{1}{2} \times .01\text{ kg} \times (300\text{ m/sec})^2$$

$$= 450\text{ joules}$$

This is enough to allow a bullet to penetrate a plank. The rest mass energy $E_0$ of the bullet is

$$E_0 = m_0c^2$$

$$= .01\text{ kg}\left(3 \times 10^8\text{ m/sec}\right)^2$$

$$= 9 \times 10^{14}\text{ joules}$$

This is the amount of energy released in a moderate-sized atomic bomb.

**Exercise 3**

For the preceding example of a 10 gram bullet:

a) at 10 cents per kilowatt hour, what is the value of the bullet's kinetic energy?

b) what is the value of its rest energy?

c) how fast would the bullet be traveling if it had twice as much kinetic energy?

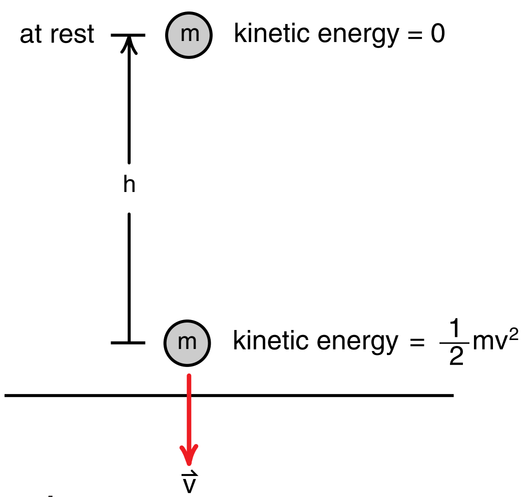



**Figure 1**
***Falling Weight. When a weight is dropped it gains kinetic energy. This kinetic energy comes from the energy we stored in the object when we lifted it up to a height h.***

## GRAVITATIONAL POTENTIAL ENERGY

Let us continue our search for terms to add to our equation for energy. Suppose that a ball of mass m is dropped from a height h above the floor, as shown in Figure (1). Immediately before the ball hits the floor, it has a rest energy $m_0c^2$, and a kinetic energy $1/2m_0v^2$. Immediately before the ball was dropped, however, it had the same rest energy $m_0c^2$ but no kinetic energy. Where did the kinetic energy that it possessed just before it hit the floor come from?

If we were observant, we might have noted that some effort was needed to lift the ball from the floor to a height h. As the ball was lifted a new kind of energy was being stored. This new form of energy, which was released when the ball was dropped, is called ***gravitational potential energy***. When it is included, our equation for energy becomes

$$E_{total} = m_0c^2 + \frac{1}{2}m_0v^2 + \begin{array}{l}\text{gravitational}\\ \text{potential}\\ \text{energy}\end{array} \qquad (8)$$

To find the formula for the gravitational potential energy, we will assume that energy is conserved and that the total energy of the ball, immediately before it is released, is equal to the total energy of the ball immediately before it hits the ground.

When a ball is dropped from a height h, it accelerates downward with a constant acceleration g until it hits the floor. Thus we can use the constant acceleration formulas (see Appendix 1 in Chapter 4.)

$$s = v_it + \frac{1}{2}at^2$$

$$v_f = v_i + at$$

with $v_i = 0$, a = g, and s = h we get

$$h = \frac{1}{2}gt^2 \qquad (12)$$

$$v_f = gt \qquad (13)$$

Substituting $t = v_f / g$ from Equation 13 into Equation (12) gives

$$h = \frac{1}{2}g\left(\frac{v_f^2}{g^2}\right) = \frac{v_f^2}{2g}$$

$$\frac{1}{2}v_f^2 = gh \qquad (14)$$

Multiplying Equation 14 through by $m_0$ gives

$$\frac{1}{2}m_0 v_f^2 = m_0 gh \qquad (15)$$

Suppose that we use $m_0gh$ as the formula for gravitational potential energy. (The greater h, the higher we have lifted the ball, the more potential energy we have stored in it.)

$$m_0gh = \left\{\begin{array}{l}\text{formula for}\\ \text{gravitational}\\ \text{potential energy}\end{array}\right. \quad \textit{near the surface of the earth} \qquad (16)$$

Before the ball is released, its total energy is in the form of rest energy and gravitational potential energy

$$E_{total}\left(\begin{array}{c}\textit{before}\\ \textit{release}\end{array}\right) = m_0c^2 + m_0gh \qquad (17)$$

Just before the ball hits the floor, where it has kinetic energy but no potential energy (since h = 0), the total energy is

$$E_{total}\left(\begin{array}{c}\textit{just before}\\ \textit{hitting floor}\end{array}\right) = m_0c^2 + \frac{1}{2}m_0 v_f^2 \qquad (18)$$

At first, Equations 17 and 18 for total energy look different; but since $1/2m_0v_f^2 = m_0gh$ (Equation 15), they give the same numerical value for the ball's total energy. Thus, we conclude that we have chosen the correct formula for calculating gravitational potential energy.

**Exercise 4**

Call $v_2$ the speed of the ball when it has fallen halfway to the floor.

(a) Explain why the ball's total energy, when it has fallen halfway to the floor, is

$$E_{total}\left(\begin{array}{c}\textit{halfway}\\ \textit{down}\end{array}\right) = m_0c^2 + \frac{1}{2}m_0v_2^2 + m_0g\left(\frac{h}{2}\right)$$

(b) Calculate $v_2$ (just as we calculated $v_f$ ) and show that the total energy of the ball when halfway down is the same as when it was released, or just before it hit the floor.

**Exercise 5**

Show that the formula for gravitational potential energy has the dimensions of joules (in the MKS system) and ergs (in the CGS system).

**Exercise 6**

What is the gravitational potential energy (in joules and ergs) of a 100–gm ball at a height of 2 meters above the floor? (Measure h starting from the floor.)

What happens to the energy after the ball has hit the floor and is lying at rest? At this point, it no longer has kinetic energy or gravitational potential energy. Now what should we add to our equation to maintain conservation of energy? In this case, we have to look "under the rug," in the "dirty water," and "out the window" all at once. When the ball hit the floor, we heard a thump; thus, some of the ball's energy has been dissipated as sound energy. We find that there is a dent in the floor; hence we know that some of the energy has gone into rearranging the molecules in that part of the floor. Also, because the bottom of the ball and the floor underneath became slightly warmer after the ball hit the floor, we conclude that some of the energy was converted into heat energy. (In some collisions, such as when a mining pick strikes a stone, we see what looks like a spark, which shows us that some of the kinetic energy has been changed into radiant energy, or light.)

After the ball hits the floor, the formula for total energy becomes as complicated as

$$E_{total} = m_0c^2 + \frac{1}{2}m_0v^2 + m_0gh + \text{sound energy} + \text{energy to cause a dent} + \text{heat energy} + \text{light energy} \qquad (19)$$

Because energy can appear in so many forms that are often difficult to detect, it was not until many years after Newton that conservation of energy was established as a general law. The law of conservation of energy is used to solve only those problems where very little energy "escapes" in a form that is difficult to detect. In a complicated collision problem we can calculate only how much energy is "lost," that is, changed to other forms of energy.

On an atomic scale, however, we do not have to think of energy as being "lost" because the various forms of energy are more easily detected. For example, we will see in Chapter 17 that the heat energy and sound energy are primarily the kinetic energy of the atoms and molecules; thus, these do not appear as separate forms of energy. It is on this small scale that the law of conservation of energy may be most accurately verified.

On the other hand, if we can neglect the effects of friction and air resistance, the law of conservation of energy can be used to solve mechanics problems that would otherwise be difficult to solve. We will illustrate this with two examples in which gravitational potential energy $m_0gh$ is converted into kinetic energy $1/2m_0v^2$ and vice versa.

### Notation

Since our discussion for the remainder of this chapter will deal with objects moving at speeds much less than the speed of light, objects whose mass m is very nearly equal to the rest mass $m_0$, we will stop writing the subscript 0 for the rest mass. With this notation, our formulas for kinetic energy and gravitational potential energy are simply $1/2mv^2$ and mgh. Only when we discuss objects like atomic particles whose speeds can become relativistic, will we be careful to distinguish the rest mass $m_0$ from the total mass m.

## Example 2

Consider a simple pendulum consisting of a ball swinging on the end of a string, as shown in Figure (2). When the ball is released from a height h it has a potential energy $m_0gh$. As the ball swings down toward the bottom, h decreases and the ball loses potential energy but gains kinetic energy. At the bottom the original potential energy mgh has been entirely converted into kinetic energy $1/2mv^2$. Then the ball climbs again, gaining potential energy but losing kinetic energy.

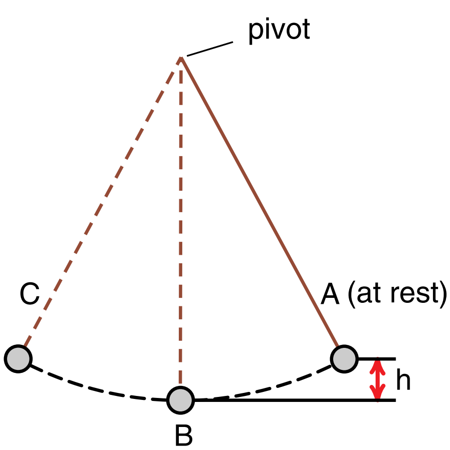



**Figure 2**
***Application of conservation of energy to pendulum motion. The speed at B can be found by equating the kinetic energy at B $(1/2mv^2)$ to the potential energy lost in going from A to B (mgh).***

Finally, at position C, the ball has swung back up to a height h and all the kinetic energy has been changed to potential energy. The ball stops momentarily at position C, and the swing is reversed. Eventually, however, the pivot becomes warm and air currents are set up by the swinging pendulum; thus, the pendulum itself gradually loses energy and finally comes to rest.

As long as we can neglect air resistance and friction in the pivot we can use the conservation of energy equation to calculate the speed of the ball at position B. Before the ball is released

$$E_{\text{total A}} = m_0c^2 + mgh$$

At position B, where h = 0

$$E_{\text{total B}} = m_0c^2 + \frac{1}{2}mv_B^2$$

If energy is conserved

$$E_{\text{total A}} = E_{\text{total B}}$$

$$\cancel{m_0c^2} + mgh = \cancel{m_0c^2} + \frac{1}{2}mv^2$$

Note that since $m_0c^2$ did not change, it does not enter into this calculation. Here we could apply the conservation of energy equation without considering the rest energy. We now have

$$mgh = \frac{1}{2}mv_B^2$$

$$v_B^2 = 2gh$$

$$v_B = \sqrt{2gh}$$

## Example 3

It should be noted that we are able to calculate the speed of the ball in the preceding example without an analysis of the forces involved. An even more striking example of conservation of energy that would be nearly impossible to analyze in terms of forces is that of a skier traveling down a very icy hill. If he is not an experienced skier, he may not know how to dissipate some of his kinetic energy as heat and sound by scraping the edges of his skis against the ice. If he is not able to dissipate energy, then no matter how he turns, no matter how twisted a path he takes, when he reaches bottom, all his potential energy $m_0gh$ will have been converted to kinetic energy $1/2m_0v^2$, in which case his speed at the bottom of the hill will be $\sqrt{2gh}$ . To see why an inexperienced skier should not try icy hills, consider that if the hill has a 500–ft rise, his speed at the bottom will be 179 ft/sec or 122 mi/hr. This result is computed not from the details of the skier's path, but from the knowledge that he was not able to dissipate energy. As we mentioned at the beginning of the chapter, the conservation of energy is one of the general principles of mechanics that can be applied successfully without knowing all the details involved in the physical situation.

---

**Exercise 7**

A car coasts along a road that leads from the top of a 300–ft–high hill, down through a valley, and up over a 200 ft high hill. Assume that the car does not dissipate energy through friction and air resistance.

(a) If the car starts at rest from atop the higher hill, how fast will it be traveling when it reaches the top of the lower hill ($g = 32\text{ ft/sec}^2$) ?

(b) If the car is initially moving at 80 ft/sec (55 mi/hr) when it starts coasting at the top of the higher hill, how fast will the car be moving when it reaches the top of the lower hill?

---

## WORK

Let us take another look at the example where we dropped a ball of mass m from a height h above the floor as shown in Figure (3). At the height h, the ball had a gravitational potential energy mgh. Just before hitting the floor, all this gravitational potential energy had been converted to kinetic energy $1/2\ mv^2$. We know that the ball speeded up, accelerated, because gravity was exerting a downward force mg on the ball as it fell.

There appears to be a coincidence in this example. Gravity pulls down on the ball with a force of magnitude mg, the ball falls a distance h, and the ball gains a kinetic energy equal to $(mg)\times h$. In this example the energy that gravity supplies to the ball by pulling down on it is equal to the gravitational force (mg) times the distance h over which the force acted. Is this a coincidence, or does this example provide a clue as to the way in which forces supply energy?

In this case, where we have a constant force mg, and the ball moves in the direction of the force for a distance h, the increase in energy is the force times the distance.

In more general examples, however, the situation can be more complex. If the object is not moving in the direction of the force, then only the component of the force in the direction of motion adds energy to the object. And if the force is not constant, we have to break the problem into many small steps, and calculate the energy gained in each step. We shall see that calculus provides powerful techniques to handle these situations.

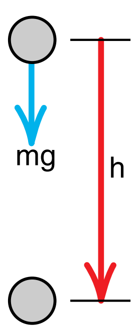



**Figure 3**
***A ball, subject to a gravitational force mg, falling a distance h, gains a kinetic energy mgh.***

We will begin the discussion with the introduction of a new term which we will call ***work***. In some ways this is an unfortunate choice of a word, for everyone has their own idea of what "work" is, and it seldom coincides with the physicist's definition. In the physicist's definition, ***a force does work on an object when it adds energy to the object***. More explicitly, the work a force does is equal to the energy that the force supplies. In the case of the falling ball the gravitational force supplied an amount of energy mgh, therefore that is the work that the gravitational force did as the ball fell.

$$\left.\begin{array}{l}\text{work done by the}\\ \text{force of gravity}\\ \text{as the ball fell}\end{array}\right\} = mgh \qquad (20)$$

From Equation (20), we see that for the case where we have a constant force, and the object moves in the direction of the force, the work done is equal to the magnitude of the force times the distance moved.

$$\boxed{\text{Work} = \text{Force} \times \text{Distance}} \qquad (21)$$

*If the force is constant and the distance is in the direction of the force*

**Exercise 8**

Show that force times distance has the same dimensions as energy. (Get the dimensions of energy from $E = mc^2$.)

As the first complication, or correction to our definition of work, suppose that the force is not in the same direction as the motion. Suppose, for example, a hockey puck slides for a distance S along frictionless ice as shown in Figure (4). During this motion a gravitational force mg is acting and the puck moves a distance S. But the puck coasts along at constant speed; it does not gain any energy at all. In this case the gravitational force does no work.

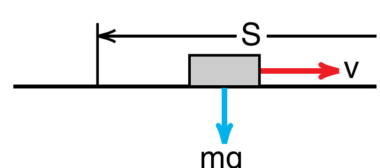



**Figure 4**
***The force of gravity does no work on the sliding hockey puck.***

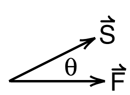
The problem with the hockey puck example is that the gravitational force is down and the motion is sideways. In this case the – y directed gravitational force has no component along the x directed motion of the puck. In order for the puck to gain energy, it must accelerate in the x direction, but there is no x component of force to produce that acceleration.

Now let us consider an example where the force is acting opposite to the direction of motion. If we throw a ball up in the air, the ball starts out with the kinetic energy $1/2\ mv_0^2$ that we gave it. As the ball rises, gravity acts against the motion of the ball and removes kinetic energy. When the ball has risen to a height h given by $mgh = 1/2\ mv_0^2$, all the kinetic energy is gone and the ball stops. The ball has reached the top of the trajectory. This example tells us that when the force is directed opposite to the direction of motion, the work is negative—the force removes rather than adds energy.

## The Dot Product

This is where our discussion has lead so far. We have a quantity called "work" which is a form of energy. It is the energy supplied by a force acting on a moving object. Now energy, given by formulas like $E = mc^2$, is a scalar quantity; it is a number that does not point anywhere. But our formula for *work = force times distance* involves two vectors, the force $\vec{F}$ and the distance $\vec{S}$. What mathematical way can we combine the two vectors $\vec{F}$ and $\vec{S}$ to get a number for the work W? One possibility, that we discussed back in the chapter on vectors, is the scalar or dot product.

$$\boxed{W = \vec{F}\cdot\vec{S}} = |\vec{F}|\,|\vec{S}|\cos\theta \qquad (22)$$




Mathematically the dot product turns the vectors $\vec{F}$ and $\vec{S}$ into a scalar number W. Let us see if $W = \vec{F}\cdot\vec{S}$ is the correct formula for work. If $\vec{F}$ and $\vec{S}$ are in the same direction, $\theta = 0°$, $\cos\theta = 1$, and we get

$$W = \vec{F}\cdot\vec{S} = |\vec{F}|\,|\vec{S}|\cos\theta = |\vec{F}|\,|\vec{S}|$$

Applied to the case of a falling ball, $|\vec{F}| = mg$, $|\vec{S}| = h$ and we get $W = mgh$ which is correct.

When we throw the ball up, the angle between the downward force and upward motion is $\theta = 180°$, $\cos\theta = -1$, and we get

$$W = \vec{F}\cdot\vec{S} = |\vec{F}|\,|\vec{S}|\cos\theta$$
$$= mgh(-1) = -mgh$$

We now predict that gravity is taking energy from the ball, which is also correct.

Finally, in the case of a hockey puck, the angle $\theta$ between the – y directed force and the x directed motion is 90°. We have $\cos\theta = 0$, so that $\vec{F}\cdot\vec{S} = 0$ and the gravitational force does no work. Again the formula $W = \vec{F}\cdot\vec{S}$ works.

---

**Exercise 9**

A frictionless plane is inclined at an angle $\theta$ as shown in Figure (5). A hockey puck initially at a height h above the ground, slides down the plane. When the puck gets to the bottom, it has moved a distance $S = h/\cos\theta$ as shown. (This comes from $h = S\cos\theta$)

a) Verify the formula $S = h/\cos\theta$ for the two cases $\theta = 0$ and $\theta = 90°$. I.e., what are the values for $h/\cos\theta$ for these two cases, and are the answers correct?

b) Show that the work $W = \vec{F}_g\cdot\vec{S}$, done by the gravitational force as the puck slides down the plane, is mgh no matter what the angle $\theta$ is.

c) Explain the result of part (b) from the point of view of conservation of energy.

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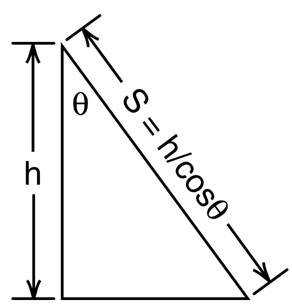


**Figure 5**
***Diagram for Exercise 9.***

## Work and Potential Energy

In the discussion of energy, physicists tend to use a lot of words like *work, potential energy, kinetic energy*, etc. What we are doing is building a conceptual picture to help us organize a number of physical phenomena and related mathematical equations. You will find that when you see this picture, are familiar with the "jargon", these concepts become easy to use and powerful in their applications. Much of this chapter is to introduce the jargon and develop the picture.

The ideas of work and potential energy are closely related and play critical roles in the picture of energy. Let us discuss some examples simply from the point of view of getting used to the jargon.

Suppose I pick a ball of mass m off the floor and slowly lift it up to a height h. While lifting the ball, I have to just barely overcome the downward gravitational force mg. Therefore I exert an upward directed force of magnitude mg, and I do this for a distance h. Since my upward force and the upward displacement are in the same direction, the work I do, call it $W_{me}$, is my force mg times the distance h, or $W_{me} = mgh$. Using the ideas of potential energy discussed earlier, we can say that all the energy $W_{me} = mgh$ that I supplied lifting the ball went into gravitational potential energy mgh.

While I was lifting the ball, gravity was pulling down. The downward gravitational force and the upward displacement were in opposite directions and therefore the work done by the gravitational force was negative. While we are storing gravitational potential energy, gravity does negative work. When we let go of the ball, gravity releases potential energy by doing positive work.

Let us consider another example where we store potential energy by doing work against a force. Suppose I tie one end of a spring to a post and pull on the other end as shown in Figure (6). As I stretch the spring, I am exerting a force $\vec{F}_{me}$ and moving the end of the spring in the same direction. Therefore I am doing positive work on the spring, and this energy is stored in what we can call the "elastic potential energy" of the stretched spring. (We know that a stretched spring has some form of potential energy, for a stretched spring can be used to launch a ball up into the air.)

## Non-Constant Forces

Our example above, of storing energy in a spring by stretching it, introduces a new complication. We cannot calculate the work I do $W_{me}$ in stretching the spring by writing $W_{me} = \vec{F}_{me} \cdot \vec{S}$. The problem is that, the farther I stretch the spring, the harder it pulls back (Hooke's law). If I slowly pull the spring out, I have to apply an increasingly stronger force. If we try to use the formula $W_{me} = \vec{F}_{me} \cdot \vec{S}$, the problem is what value of $\vec{F}_{me}$ to use. Do we use the weak $\vec{F}_{me}$ at the beginning of the pull, the strong one at the end, or some average value.

We could use an average value, but there is a more general way to calculate the work I do. Suppose I wish to pull the spring from an initial position $x_i$ to a final position $x_f$. Imagine that I break this span from $x_i$ to $x_f$ into a bunch of small intervals of width $\Delta x$, ending at points labeled $x_0$, $x_1$, ... $x_n$ as shown in Figure (7). During each small interval the spring force does not change by much, and I can stretch the spring through that interval by exerting a force equal to the strength of

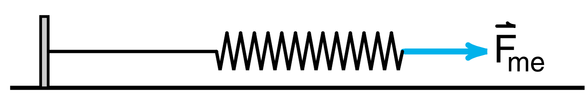


**Figure 6**
***Doing work on a spring.***

the spring force at the end of the interval. For example in stretching the spring from position $x_0$ to $x_1$, I apply a force of magnitude $F_s(x_1)$ for a distance $\Delta x$ and therefore do an amount of work

$$(\Delta W_{me})_1 = F_s(x_1)\Delta x$$

To get out to position $x_2$, I increase my force to $F_s(x_2)$ and apply that force over another interval $\Delta x$ to do an amount of work

$$(\Delta W_{me})_2 = F_s(x_2)\Delta x$$

If I keep repeating this process until I reach the final position $x_f$, the total amount of work I have done is

$$\begin{aligned}(W_{me})_{total} &= (\Delta W_{me})_1 + (\Delta W_{me})_2 + \ldots \\ &\quad + (\Delta W_{me})_n \\ &= F_s(x_1)\Delta x + F_s(x_2)\Delta x + \ldots \\ &\quad + F_s(x_n)\Delta x \\ &= \sum_{i=1}^{n} F_s(x_i)\Delta x \qquad (23)\end{aligned}$$

In Equation 23, we still have an approximate calculation as long as the intervals $\Delta x$ are of finite size. We get an exact calculation of the work I do if we take the limit as $\Delta x$ goes to zero, and the number of intervals goes to infinity. In that limit, the right side of Equation 23 becomes the definite integral of $F_s(x)$ from the initial position $x_i$ to the final position $x_f$ :

$$(W_{me})_{total} = -\int_{x_i}^{x_f} F_s(x)\,dx \qquad (24)$$

The statement of the work we did, Equation 24, can be written more formally by noting that the spring force $\vec{F}_s(x)$ is actually a vector which points opposite to the direction I pulled the spring. In addition, we should think of each $\Delta x$ or dx as a small vector displacement $\vec{\Delta x}$ or $\vec{dx}$ in the direction I pulled. Since my force was directed opposite to $\vec{F}_s$, the work I did during each interval $\vec{dx}$ can be written as the dot product

$$dW_{me} = \vec{F}_{me} \cdot \vec{dx} = -\vec{F}_s \cdot \vec{dx}$$

and the formula for the total work I did becomes

$$\boxed{(W_{me})_{total} = -\int_{x_i}^{x_f} \vec{F}_s \cdot \vec{dx}} \qquad (25)$$

Equation 25 is more general but a bit clumsier to use than 24. To use Equation 25, we would first note that I was pulling along the x axis, and thus $\vec{dx} = dx$ . Then I would note that the spring force was opposite to the direction I was pulling, so that $-\vec{F}_s(x) \cdot \vec{dx} = +F_s(x)dx$

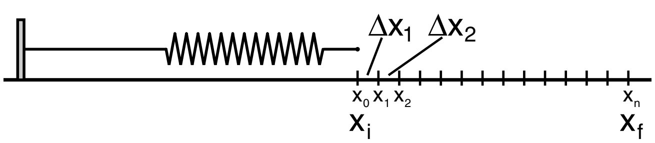


**Figure 7**
***I can stretch the spring through a series of small intervals of length Δx. In each interval I apply a constant force that is just strong enough to get the spring to the end of the interval.***

where $F_s(x)$ is the formula for the strength of the spring force. That gets me back to Equation (24) and the problem of evaluating the definite integral.

## Potential Energy Stored in a Spring

Springs are useful in physics demonstrations and problems because of the simple force law (Hooke's law) which is quite accurately obeyed by real springs. In our study of the motion of a ball on the end of a spring in Chapter 9, we saw that the formula for the strength of the spring force was

$$F_s = K(S - S_0) \qquad (9\text{–}6)$$

where S is the length of the spring and $S_0$ the unstretched length (the length at which $F_s$ goes to zero in Figure 9–4).

We can simplify the spring force formula, get rid of the $S_0$, by considering a situation where an object is held in an equilibrium position by spring forces. Suppose for example we have a cart on an air track with springs connecting the cart to each end of the track as shown in Figure (8). Mark the center of the cart with an arrow, and choose a coordinate system where x = 0 is at the equilibrium position as shown in Figure (8a).

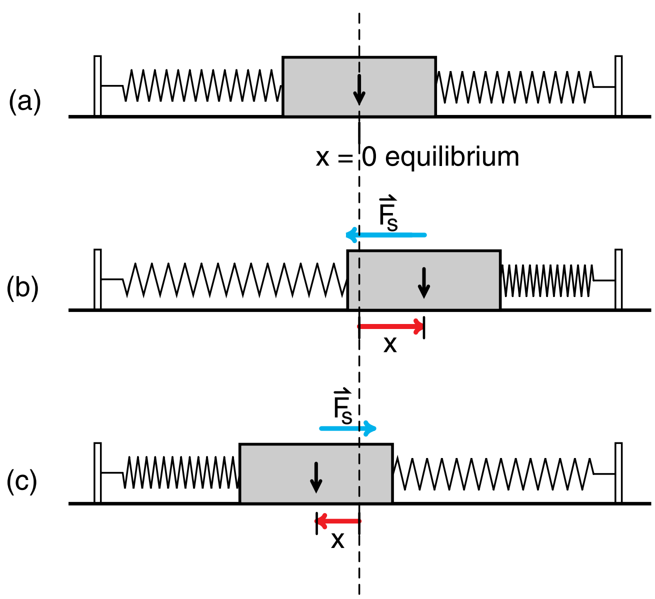


**Figure 8**
***The spring force $\vec{F}_s$ is always opposite to the displacement x. If the spring is displaced right, $\vec{F}_s$ points left, and vice versa.***

With this setup, the spring force is always a restoring force that is pushing the cart back to the equilibrium position x = 0. If we give the cart a positive displacement as in Figure (8b), we get a left directed or negative spring force. A negative displacement shown in (8c) produces a right directed or positive spring force. And to a high degree of accuracy, the strength of the spring force is proportional to the magnitude of the displacement from equilibrium.

All of these results can be described by the formula

$$\boxed{F_s(x) = -Kx} \qquad (26)$$

where the minus sign tells us that a positive displacement x produces a negative directed force and vice versa. There is no $S_0$ or $x_0$ in Equation 26 because we chose x = 0 to be the equilibrium position where $F_s = 0$. Equation 26 is what one usually finds as a statement of Hooke's law, and K is called the spring constant.

Equation 26 allows us to easily calculate the potential energy stored in the springs. If I start with the cart at rest at the equilibrium position as shown in Figure (8a), and pull the cart to the right a distance $x_f$, the work I do is

$$W_{me} = \int_{x=0}^{x=x_f} F_{me}\, dx = \int_{x=0}^{x=x_f} (-F_s)\, dx$$

$$= \int_{x=0}^{x=x_f} Kx\, dx \qquad (27)$$

where I have to exert a force $F_{me} = -F_s$ to stretch the spring.

In Equation 27, the constant K can come outside the integral, we are left with the integral of xdx which is $x^2/2$, and we get

$$W_{me} = K\int_{x=0}^{x=x_f} x\,dx = K\frac{x^2}{2}\bigg|_0^{x_f} = K\frac{x_f^2}{2}$$

Noting that all the work I do is stored as "*elastic potential energy of the spring*", we get the formula

$$\boxed{\text{Spring potential energy} = K\frac{x^2}{2}} \qquad (28)$$

In Equation 28, we replaced $x_f$ by x since the formula applies to any displacement $x_f$ I choose.

---

**Exercise 10**

If you pull the cart of Figure (8) back a distance $x_f$ from the equilibrium position and let go, all the potential energy you stored in the cart will be converted to kinetic energy when the cart crosses the equilibrium position x = 0. Use this example of conservation of energy to calculate the speed v of the cart when it crosses x = 0. (Assume that you release the cart from rest.)

---

Exercise 10, which you should have done by now, illustrates one of the main reasons for bothering to calculate potential energy. It is much easier to predict the speed of the ball using energy conservation than it is using Newton's second law. We can immediately find the speed of the ball by equating the kinetic energy at x = 0 to the potential energy at $x = x_f$ where we released the cart. To make the same prediction using Newton's second law, we would have to solve a differential equation and do a lot more calculation.

---

**Exercise 11**

With a little bit of cleverness, we can use energy conservation to predict the speed of the cart at any point along the air track. Suppose you release the cart from rest at a distance $x_f$, and want to know the cart's speed at, say, $x_f/2$. First calculate how much potential energy the cart loses in going from $x_f$ to $x_f/2$, and then equate that to the kinetic energy $1/2\,mv^2$ that the cart has gained at $x_f/2$.

---

## WORK ENERGY THEOREM

The reason that it is easier to apply energy conservation than Newton's second law is that when we have a formula for potential energy, we have already done much of the calculation. We can illustrate this by deriving what is called the "Work Energy Theorem" where we use Newton's second law to derive a relation between work and kinetic energy. We will first derive the theorem for one dimensional motion, and then see that it is easily extended to motion in three dimensions.

Suppose a particle is moving along the x axis as shown in Figure (9). Let a force $F_x(x)$ be acting on the particle. Then by Newton's second law

$$F_x(x) = ma_x(x) = m\frac{dv_x(x)}{dt} \tag{29}$$

Multiplying by dx and integrating to calculate the work done by the force $F_x$, we get

$$\int_i^f F_x(x)dx = m\int_i^f \frac{dv_x(x)}{dt}dx \tag{30}$$

In Equation (30), we are integrating from some initial position $x_i$ where the object has a speed $v_{xi}$, to a position $x_f$ where the speed is $v_{xf}$.

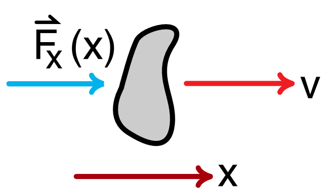


**Figure 9**
***An x directed force acting on a particle moving in the x direction.***

The next step is a standard calculus trick that you may or may not remember. We will first move things around a bit in the integral on the right side of Equation 30:

$$m\int_i^f \frac{dv_x}{dt}dx = m\int_i^f dv_x\left(\frac{dx}{dt}\right) \tag{31}$$

Next note that $dx/dt = v_x$, the x component of the velocity of the particle. Thus the integral becomes

$$m\int_i^f dv_x\left(\frac{dx}{dt}\right) = m\int_{v_i}^{v_f} v_x dv_x \tag{32}$$

After this transformation, we can do the integral because everything is now expressed in terms of the one variable $v_x$. Using the fact that the integral of $v_x dv_x$ is $v_x^2/2$, we get

$$\begin{aligned} m\int_{v_i}^{v_f} v_x dv_x &= m\frac{v_x^2}{2}\Bigg|_{v_i}^{v_f} \\ &= \frac{1}{2}mv_{fx}^2 - \frac{1}{2}mv_{ix}^2 \end{aligned} \tag{33}$$

Using Equations (31) through (33) in Equation (30) gives

$$\boxed{\int_{x_i}^{x_f} F_x(x)dx = \frac{1}{2}mv_{fx}^2 - \frac{1}{2}mv_{ix}^2} \tag{34}$$

The left side of Equation 34 is the work done by the force $F_x$ as the particle moves from position $x_i$ to position $x_f$. The right side is the change in the kinetic energy. Equation 34 tells us that the work done by the force $F_x$ equals the change in the particle's kinetic energy. This is the basic idea of the work energy theorem.

To derive the three dimensional form of Equation 34, start with Newton's second law in vector form

$$\vec{F} = m\vec{a} \qquad (35)$$

Take the dot product of Equation 35 with $\vec{dx}$ and integrate from i to f to get

$$\int_i^f \vec{F}\cdot\vec{dx} = \int_i^f m\vec{a}\cdot\vec{dx} \qquad (36)$$

Writing

$$\vec{a}\cdot\vec{dx} = a_x dx + a_y dy + a_z dz \qquad (37)$$

we get

$$\int_i^f \vec{F}\cdot\vec{dx} = m\int_i^f \left(\frac{dv_x}{dt}dx + \frac{dv_y}{dt}dy + \frac{dv_z}{dt}dz\right) \qquad (38)$$

Following the same steps we used to get from Equation 31 to 33, we get

$$\int_i^f \vec{F}\cdot\vec{dx} = \frac{1}{2}mv_{fx}^2 - \frac{1}{2}mv_{ix}^2 + \frac{1}{2}mv_{fy}^2 - \frac{1}{2}mv_{iy}^2 + \frac{1}{2}mv_{fz}^2 - \frac{1}{2}mv_{iz}^2 \qquad (39)$$

Finally noting that by the Pythagorean theorem

$$v_i^2 = v_{ix}^2 + v_{iy}^2 + v_{iz}^2$$
$$v_f^2 = v_{fx}^2 + v_{fy}^2 + v_{fz}^2 \qquad (40)$$

we get, using (40) in (39)

$$\boxed{\int_i^f \vec{F}\cdot\vec{dx} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2} \qquad (41)$$

which is the three dimensional form of the work energy theorem.

## Several Forces

Suppose several forces $\vec{F}_1$, $\vec{F}_2$, ... are acting on the particle as the particle moves from position i to position f. Then the vector $\vec{F}$ in Equations 35 through 41 is the total force $\vec{F}_{tot}$ which is the vector sum of the individual forces:

$$\vec{F} = \vec{F}_{tot} = \vec{F}_1 + \vec{F}_2 + \ldots \qquad (42)$$

Our formula for the work done by these forces becomes

$$\int_i^f \vec{F}\cdot\vec{dx} = \int_i^f (\vec{F}_1 + \vec{F}_2 + \ldots)\cdot\vec{dx}$$

$$= \int_i^f \vec{F}_1\cdot\vec{dx} + \int_i^f \vec{F}_2\cdot\vec{dx} + \ldots \qquad (43)$$

and we see that the work done by several forces is just the numerical sum of the work done by each force acting on the object. Equation 41 now has the interpretation that ***the total work done by all the forces acting on a particle is equal to the change in the kinetic energy of the particle***.

## Conservation of Energy

To see how the work energy theorem leads to the idea of conservation of energy, suppose we have a particle subject to one force, like the spring force $\vec{F}_s$ acting on an air cart as shown in Figure (8). If the cart moves from position i to position f, then the work energy theorem, Equation 41 gives

$$\int_i^f \vec{F}\cdot\vec{dx} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \qquad (44)$$

In our analysis of the spring potential energy, we saw that if I slowly moved the cart from position i to position f, I had to exert a force $\vec{F}_{me}$ that just overcame the spring force $\vec{F}_s$, i.e., $\vec{F}_{me} = -\vec{F}_s$. When I moved the cart slowly, the work I did went into changing the potential energy of the cart. Thus the formula for the change in the cart's potential energy is

$$\left.\begin{matrix}\text{change in the}\\ \text{potential energy of}\\ \text{the cart when the}\\ \text{cart moves from}\\ \text{position i to position f}\end{matrix}\right\} = \int_i^f \vec{F}\cdot\vec{dx} = -\int_i^f \vec{F}_s\cdot\vec{dx} \qquad (45)$$

Equation 45 is essentially equivalent to Equation 25 which we derived in our discussion of spring forces.

Spring forces have the property that the energy stored in the spring depends only on the length of the spring, and not on how the spring was stretched. This means that the change in the spring's potential energy does not depend upon whether I moved the cart, or I let go and the spring moves the cart. We should remove $\vec{F}_{me}$ from Equation 45 and simply express the spring potential energy in terms of the spring force

$$\left.\begin{matrix}\text{change in spring}\\ \text{potential energy}\end{matrix}\right\} = -\int_i^f \vec{F}_s\cdot\vec{dx} \qquad (46)$$

Equation 46 says that the change in potential energy is **minus** the work done by the force on the object as the object moves from i to f. There is a minus sign because, if the force does positive work, potential energy is released or decreases. We will see that Equation 46 is a fairly general relationship between a force and it's associated potential energy.

We are now ready to convert the work energy theorem into a statement of conservation of energy. Rewrite Equation 44 with the work term on the right hand side and we get

$$0 = \left\{-\int_i^f \vec{F}_s\cdot\vec{dx}\right\} + \left\{\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2\right\} \qquad (47)$$

The term in the first curly brackets is the change in the particle's potential energy, the second term is the change in the particle's kinetic energy. Equation 47 says that the sum of these two changes is zero

$$0 = \left\{\begin{matrix}\text{change in}\\ \text{potential energy}\end{matrix}\right\} + \left\{\begin{matrix}\text{change in}\\ \text{kinetic energy}\end{matrix}\right\} \qquad (47a)$$

If we define the total energy of the particle as the sum of the particle's potential energy plus its kinetic energy, then the change in the particle's total energy in moving from position i to position f is the sum of the two changes on the right side of Equation 47a. Equation 47a says that this total change is zero, or that the total energy is conserved.

## Conservative and Non-Conservative Forces

We mentioned that the potential energy stored in a spring depends only on the amount the spring is stretched, and not on how it was stretched. This means that the change in potential energy depends only on the initial and final lengths of the spring, and not on how we stretched it. This implies that the integral

$$-\int_i^f \vec{F}_s \cdot \vec{dx}$$

has a unique value that does not depend upon how the particle was moved from i to f.

Gravitational forces have a similar property. If I lift an object from the floor to a height h, the increase in gravitational potential energy is mgh. This is true whether I lift the object straight up, or run around the room five times while lifting it. The formula for the change in gravitational potential energy is

$$\left.\begin{matrix}\text{change in}\\ \text{gravitational}\\ \text{potential energy}\end{matrix}\right\} = -\int_i^f \vec{F}_g \cdot \vec{dx}$$

$$= -\int_0^h F_{gy} dy$$

$$= -\int_0^h (-mg) dy$$

$$= mgh \tag{48}$$

Again we have the change in potential energy equal to minus the work done by the force.

Not all forces, however, work like spring and gravitational forces. Suppose I grab an eraser and push it around on the table top for a while. In this case I am overcoming the friction force between the table and the eraser, and we have $\vec{F}_{me} = -\vec{F}_{friction}$. The total work done by me as I move the eraser from an initial position i to a final position f is

$$\left.\begin{matrix}\text{work I do}\\ \text{while moving}\\ \text{the eraser}\end{matrix}\right\} = -\int_i^f \vec{F}_{me} \cdot \vec{dx}$$

$$= -\int_i^f \vec{F}_{friction} \cdot \vec{dx} \tag{49}$$

There are two problems with this example. The integrals in Equation 49 do depend on the path I take. If I move the eraser around in circles I do a lot more work than if I move it in a straight line between the two points. And when I get to position f, there is no stored potential energy. Instead all the energy that I supplied overcoming friction has probably been dissipated in the form of heat.

Physicists divide all forces in the world into two categories. Those forces like gravity and the spring force, where the integral

$$\int_i^f \vec{F} \cdot \vec{dx}$$

depends only on the initial and final positions i and f, are called "*conservative*" forces. For these forces there is a potential energy, and the formula for the change in potential energy is minus the work the force does when the particle goes from i to f.

All the other forces, the ones for which the work integral depends upon the path, are called ***non-conservative*** forces. We cannot use the concept of potential energy for non-conservative forces because the formula for potential energy would not have a unique or meaningful value. The non-conservative forces can do work and change kinetic energy, but as we see in the case of friction, the work ends up as something else like heat rather than potential energy.

It is interesting that on an atomic scale, where energy does not disappear in subtle ways like heat, we almost always deal with conservative forces and can use the concept of potential energy.

## GRAVITATIONAL POTENTIAL ENERGY ON A LARGE SCALE

In our computer analysis of satellite motion, we saw that the quantity $E_{tot}$, given by

$$E_{tot} = \frac{1}{2}mv^2 - \frac{GM_e m}{r} \qquad (50)$$

was unchanged as the satellite moved around the earth. As shown in Figure (10), m is the mass of the satellite, $\vec{v}$ its velocity, R its distance from the center of the earth, and $M_e$ is the mass of the earth. This was our first non trivial example of conservation of energy, where $1/2\ mv^2$ is the satellite's kinetic energy, and $-GM_e m/R$ must be the formula for the satellites's gravitational potential energy. Our discussion of the last section suggests that we should be able to obtain this formula for gravitational potential energy by integrating the gravitational force $|\vec{F}_g| = GM_e m/r^2$ from some initial to some final position.

Here on the surface of the earth, the formula for gravitational potential energy is mgh. This simple result arises from the fact that when we lift an object inside a room, the strength of the gravitational force $m\vec{g}$ acting on it is essentially constant. Thus the work I do lifting a ball a distance h is just the gravitational force mg times the height h. Since this work is stored as potential energy, the formula for gravitational potential energy is simply mgh.

In the case of satellite motion, however, the strength of the gravitational force was not constant. In our first calculation of satellite motion in Chapter 8, the satellite started 1.1 earth radii from the center of the earth and went out as far as r = 5.6 earth radii. Since the gravitational force drops off as $1/r^2$, the gravitational force was more than 25 times weaker when the satellite was far away, than when it was launched.

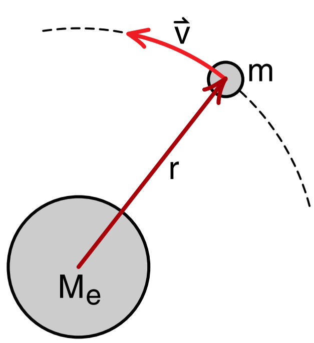


**Figure 10**
*Earth satellite.*

### Zero of Potential Energy

Another difference is that the formula mgh for a ball in the room measures changes in gravitational potential energy starting from the floor where h = 0. In a rather arbitrary way,we have defined the gravitational potential energy to be zero at the floor. This is a convenient choice for people working in this room, but people working upstairs or downstairs would naturally choose their own floors rather than our floor as the zero of gravitational potential energy for objects they were studying.

Since conservation of energy deals only with changes in energy, it does not make any difference where you choose your zero of potential energy. A different choice simply adds a constant to the formula for total energy, and an unchanging or constant amount of energy cannot be detected. The most famous example of this was the fact that a particle's rest energy $m_0c^2$ was unknown until Einstein introduced the special theory of relativity, and undetected until we saw changes in rest energy caused by nuclear reactions. In the case of the gravitational potential energy of a ball, if we use the floor downstairs as the zero of gravitational potential energy, we add the constant term $(mg)h_{floor}$ to all our formulas for $E_{tot}$ (where h floor is the distance between floors in this building). This constant term has no detectable effect.

In finding a formula for gravitational potential energy of satellites, planets, stars, etc., we should select a convenient floor or zero of potential energy. For the motion of a satellite around the earth, we could choose gravitational potential energy to be zero at the earth's surface. Then the satellite's potential energy would be positive when its distance r from the center of the earth is greater than the earth radius $r_e$, and negative should r become less than $r_e$. Such a choice would be reasonable if we were only going to study earth satellites, but the motion of a satellite about the earth is very closely related to the motion of the planets about the sun and the motion of moons about other planets. Choosing $r = r_e$ as the distance at which gravitational potential energy is zero is neither a general or particularly convenient choice.

In describing the interaction between particles, for example an electron and a proton in a hydrogen atom, the earth and a satellite, the sun and its planets, or the stars in a galaxy, the convenient choice for the zero of potential energy is where the particles are so far apart that they do not interact. If the earth and a rock are a hundred light years apart, there is almost no gravitational force between them, and it is reasonable that they do not have any gravitational potential energy either.

Now suppose that *the earth and the rock are the only things in the universe*. Even at a hundred light years there is still some gravitational attraction, so that the rock will begin to fall toward the earth. As the rock gets closer to the earth it will pick up speed and thus gain kinetic energy. It was the gravitational force of attraction that caused this increase in speed, therefore there must be a conversion of gravitational potential energy into kinetic energy.

This gives rise to a problem. The rock starts with zero gravitational potential energy when it is very far away. As the rock approaches the earth, gravitational potential energy is converted into kinetic energy. How can we convert gravitational potential energy into kinetic energy if we started with zero potential energy?

Keeping track of energy is very much a bookkeeping scheme, like keeping track of the balance in your bank account. Suppose you begin the month with a balance of zero dollars, and start spending money by writing checks. If you have a trusting bank, this works because your bank balance simply becomes negative.

In much the same way, the rock falling toward the earth started with zero gravitational potential energy. As the rock picked up speed falling toward the earth, it gained kinetic energy at the expense of potential energy. Since it started with zero potential energy, and spent some, it must have a negative potential energy balance. From this we see that if we choose gravitational potential energy between two objects to be zero when the objects are very far apart, then the potential energy must be negative when the objects are a smaller distance apart. When we think of energy conservation as a bookkeeping scheme, then the idea of negative potential energy is no worse than the idea of a negative checking account balance.

(In the analogy between potential energy and a checking account, the discovery of rest energy $m_0c^2$ would be like discovering that you had inherited the bank. The checks still work the same way even though your total assets are vastly different.)

Let us now return to Equation (50) and our formula for gravitational potential energy of a satellite

$$\left.\begin{array}{l}\text{gravitational}\\ \text{potential energy}\end{array}\right\} = -\frac{GM_em}{r} \qquad (50a)$$

First we see that if the satellite is very far away, that as r goes to infinity, the potential energy goes to zero. Thus this formula does give zero potential energy when the earth and the satellite are so far apart that they no longer interact. In addition, the potential energy is negative, as it must be if the satellite falls in to a distance r, converting potential energy into kinetic energy.

What we have to do is to show that Equation (50a) is in fact the correct formula for gravitational potential energy. We can do that by calculating the work gravity does on the satellite as it falls in from $r = \infty$ to $r = r$. This work, which would show up as the kinetic energy of a falling satellite, must be the amount of potential energy spent. Thus the potential energy balance must be the negative of this work. Since the work is the integral of the gravitational force times the distance, we have

$$\left.\begin{array}{l}\text{gravitational}\\ \text{potential energy}\\ \text{at position R}\end{array}\right\} = -\left|\int_{\infty}^{R} \vec{F}_g \cdot d\vec{r}\right|$$

$$= -\left|\int_{\infty}^{R} \frac{GMm}{r^2}\,dr\right| \qquad (51)$$

Equation 51 may look a bit peculiar in the way we have handled the signs. We have argued physically that the gravitational potential energy must be negative, and we know that it must be equal in magnitude to the integral of the gravitational force from $r = \infty$ to $r = R$. By noting ahead of time what the sign of the answer must be, we can do the integral easily without keeping track of the various minus signs that are involved. (One minus sign is in the formula for potential energy, another is the dot product since $\vec{F}_g$ points in and $d\vec{r}$ out, a third in the integral of $r^{-2}$, and more come in the evaluation of the limits. It is not worth the effort to get all these signs right when you know from a simple physical argument that the answer must be negative.)

Carrying out the integral in Equation 51 gives

$$\int_{\infty}^{R} \frac{GM_e m}{r^2}\,dr = GM_e m \int_{\infty}^{R} \frac{dr}{r^2}$$

$$= -\left.\frac{GM_e m}{r}\right|_{\infty}^{R} = GM_e m\left(\frac{1}{\infty} - \frac{1}{R}\right)$$

where we used the fact that the integral of $1/r^2$ is $-1/r$. Thus we get

$$\left|\int_{\infty}^{R} \frac{GM_e m}{r^2}\,dr\right| = \frac{GM_e m}{R}$$

As a result the gravitational potential energy of the satellite a distance R from the center of the earth is $-GM_e m/R$ as given in Equation 50a.

## Gravitational Potential Energy in a Room

Before we leave our discussion of gravitational potential energy, we should show that the formula $-GM_em/r$ leads to the formula mgh for the potential energy of a ball in a room. To show this, let us use the formula $-GM_em/r$ to calculate the increase in gravitational potential energy when I lift a ball from the floor, a distance $R_e$ from the center of the earth, up to a height h, a distance $R_e + h$ from the center of the earth, as shown in Figure (11).

We have

$$PE_{\text{at height h}} = -\frac{GM_em}{R_e + h}$$

$$PE_{\text{at floor}} = -\frac{GM_em}{R_e}$$

$$\left.\begin{matrix}\text{Increase}\\ \text{in PE}\end{matrix}\right\} = PE_{\text{at h}} - PE_{\text{at floor}}$$

$$= \left(-\frac{GM_em}{R_e + h}\right) - \left(-\frac{GM_em}{R_e}\right)$$

$$= GM_em\left(\frac{1}{R_e} - \frac{1}{R_e + h}\right) \qquad (52)$$

To evaluate the right side of Equation 52, we can write

$$\frac{1}{R_e + h} = \frac{1}{R_e}\left(\frac{1}{1 + h/R_e}\right)$$

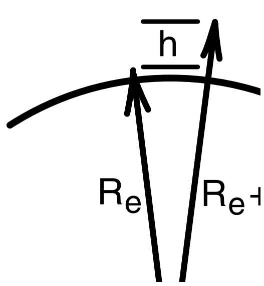


**Figure 11**
***A height h above the surface of the earth.***

Since $h/R_e$ is a very small number compared to one, we can use our small number approximation

$$\frac{1}{1+\alpha} \approx 1 - \alpha \qquad \text{if } \alpha << 1$$

to write

$$\frac{1}{1 + h/R_e} \approx 1 - \frac{h}{R_e}$$

so that

$$\frac{1}{R_e + h} \approx \frac{1}{R_e}\left(1 - \frac{h}{R_e}\right) = \frac{1}{R_e} - \frac{h}{R_e^2} \qquad (53)$$

Using Equation 53 in (52) gives

$$\left.\begin{matrix}\text{Increase}\\ \text{in PE}\end{matrix}\right\} = -GM_em\left[\frac{1}{R_e} - \left(\frac{1}{R_e} - \frac{h}{R_e{}^2}\right)\right]$$

$$= GM_em\left[\frac{h}{R_e{}^2}\right]$$

$$= \left(\frac{GM_e}{R_e{}^2}\right)mh$$

Finally noting that $GM_e/R_e{}^2 = g$, the acceleration due to gravity at the surface of the earth, we get

$$\boxed{\left.\begin{matrix}\text{Increase}\\ \text{in PE}\end{matrix}\right\} = mgh}$$

which is the expected result.

## SATELLITE MOTION AND TOTAL ENERGY

Consider a satellite moving in a circular orbit about the earth, as shown in Figure (12). We want to calculate the kinetic energy, potential energy, and total energy (sum of the kinetic and potential energy) for the satellite. To find the kinetic energy, we analyze its motion, using Newton's laws. The only force acting on the satellite is the gravitational force $\vec{F}_g$ given by

$$\left|\vec{F}_g\right| = \frac{GMm}{r^2} \qquad \textit{$\vec{F}_g$ directed toward the earth}$$

where we now let M = mass of the earth and m = mass of the satellite. Since the satellite is moving at constant speed v in a circle of radius r, its acceleration is $v^2/r$ toward the center of the circle

$$|\vec{a}| = \frac{v^2}{r} \qquad \textit{$\vec{a}$ directed toward the earth}$$

Since $\vec{a}$ and $\vec{F}_g$ are in the same direction, by Newton's second law

$$\left|\vec{F}_g\right| = m|\vec{a}|$$

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

From this last equation we find that the kinetic energy $1/2mv^2$ of the satellite is

$$\frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{r} \qquad \textit{kinetic energy}$$

The kinetic energy, as always, is positive.

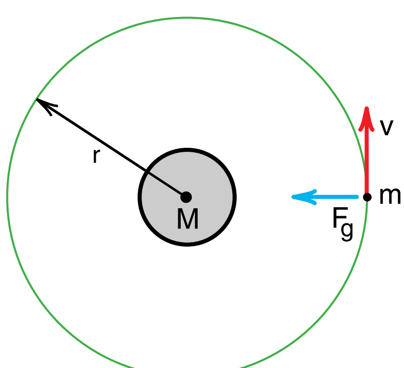


**Figure 12**
***Satellite in a circular orbit.***

The gravitational potential energy of the satellite is always negative. Since the satellite is a distance r from the center of the earth, its potential energy is

$$\text{potential energy} = -\frac{GMm}{r}$$

The total energy of the satellite is

$$E_{total} = \text{kinetic energy} + \text{potential energy}$$

$$= \frac{1}{2}\frac{GMm}{r} + \left(-\frac{GMm}{r}\right)$$

$$E_{total} = -\frac{1}{2}\frac{GMm}{r} \qquad (54)$$

The total energy of a satellite in a circular orbit is negative.

Now consider a satellite in an elliptical orbit. In particular, suppose that the orbit is an extended ellipse, as shown in Figure (13). At apogee, the farthest point from the earth, the satellite is moving very slowly (explain why by using Kepler's law of equal areas). For all practical purposes, the satellite drifts out, stops at apogee, then falls back toward the earth. At apogee, the satellite has *almost no kinetic energy*; at this point its total energy is nearly equal to its negative potential energy

$$E_{total} = -\frac{GMm}{r_{apogee}}$$

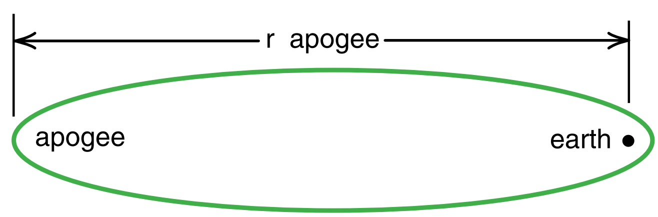


**Figure 13**
***Satellite in a very eccentric orbit. By Kepler's law of equal areas, a satellite with the above orbit would almost be at rest at apogee.***

Since the total energy is conserved, $E_{total}$ remains negative throughout the orbit. If similar satellites are placed in different orbits, the one that goes out the farthest (has the greatest $r_{apogee}$) is the one with the least negative total energy, but all the satellites in elliptical orbits will have a negative total energy.

Suppose an extra powerful rocket is used and a satellite is launched with a positive total energy. In such a case, the positive kinetic energy must always exceed the negative potential energy. No matter how far out the satellite goes, headed for apogee, it will always have some positive kinetic energy to carry it out farther.

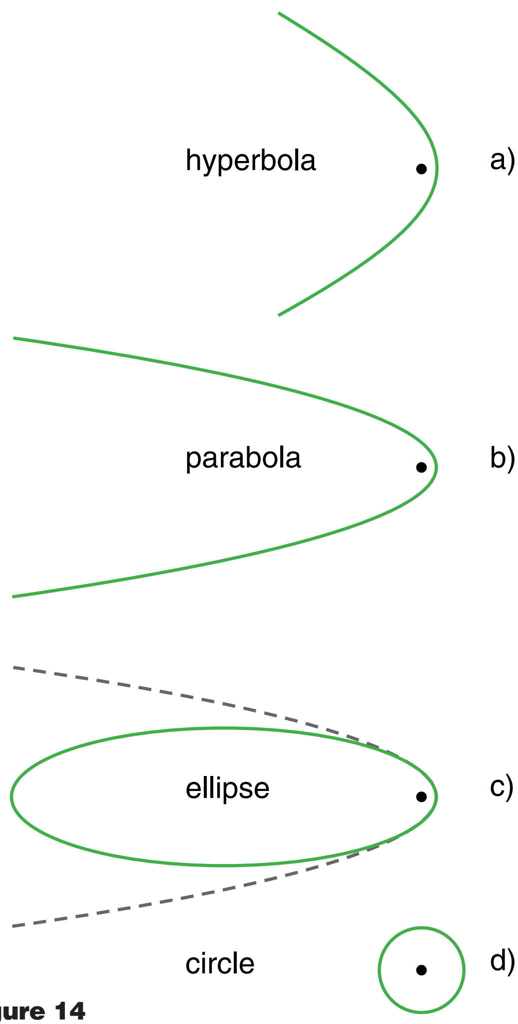


**Figure 14**
***a) Hyperbolic orbit of comet with positive total energy.***
***b) Parabolic orbit of comet with zero total energy.***
***c) Elliptical orbit of comet with slightly negative total energy. (Dashed lines show parabolic orbit for comparison.)***
***d) Nearly circular orbits of the tightly bound (large negative energy) planets.***

Even at enormous distances, where the negative potential energy $-GMm/r$ is about zero, some kinetic energy would still remain, and the satellite would escape from the earth!

By choosing potential energy to be zero when the satellite is very far out, the total energy becomes a meaningful number in itself. If the total energy is negative, the satellite will remain bound to the earth; it does not have sufficient energy to escape. If a satellite launched with positive total energy, it must escape since the negative gravitational potential energy is not sufficiently great to bind the satellite to the earth. If the satellite's total energy is zero, it barely escapes.

The orbits of comets about the sun are interesting examples of orbits of different total energies. It can be shown that when a *satellite's* total energy is positive, its orbit will be in the shape of a ***hyperbola***, which is an open-ended curve, as shown in Figure (14a). In this orbit the comet has a positive total energy and never returns.

If the total energy of the comet is zero, the orbit will be in the shape of an open curve, called a ***parabola*** (Figure 14b). A comet in this kind of orbit will not return either.

When the comet's total energy is slightly less than zero, it must return to the sun. In this situation the comet's orbit is an ellipse, even though it may be a very extended ellipse. A comparison of an extended ellipse and a parabola is shown in Figure (14c). From this figure we can see that near the sun there is not much difference in the motion of a comet with zero or slightly negative total energy. The difference can be seen at a great distance, where the zero-energy comet continues to move away from the sun, but the slightly negative-energy comet returns.

The circular, or nearly circular, motion of the planets is a limiting case of elliptical motion. The small circular orbits (Figure 14d) are occupied by planets that have large negative total energies. Thus the planets are tightly bound to the sun.

## Example 4 Escape Velocity

At what speed must a shell be fired from a super cannon in order that it escapes from the earth? Does it make any difference at what angle the shell is fired, so long as it clears all obstructions? (Neglect air resistance.)

**Solution**: If the shell is fired at a sufficiently great initial speed so that its total energy is positive, it will eventually escape from the earth, regardless of the angle at which it is fired (so long as it clears obstructions). To calculate the minimum muzzle velocity at which the shell can escape, we will assume that the shell has zero total energy, so that it barely escapes. When $E_{total} = 0$ we have just after the shell is fired

$$0 = \frac{1}{2}mv^2 - \frac{GM_em}{r_e}$$

which gives

$$v^2 = \frac{2GM_e}{r_e} \tag{55}$$

Putting in numbers

$$\begin{aligned} G &= 6.67 \times 10^{-8}\ cm^3/gm\ \ sec^2 \\ M_e &= 5.98 \times 10^{27}\ gm \\ r_e &= 6.38 \times 10^8\ cm \end{aligned}$$

we get

$$\begin{aligned} v^2 &= \frac{2\times\left(6.67\times10^{-8}\ cm^3/gm\ \ sec^2\right)\left(5.98\times10^{27}\ gm\right)}{6.38\times10^8\ cm} \\ &= \frac{2\times6.67\times5.98}{6.38}\ \frac{10^{-8}\times10^{27}}{10^8}\ cm^3gm/gm\ cm\ sec^2 \\ &= 1.25\times10^{12}\ cm^2/sec^2 \end{aligned}$$

$$v_{escape} = 1.12\times10^6\ cm/sec$$

Converting this to more recognizable units, such as mi/sec, we have

$$\begin{aligned} v_{escape} &= 1.12\times10^6\ cm/sec \times \frac{1}{1.6\times10^5\ cm/mi} \\ &= 7\ mi/sec \quad (11.2\ km/sec) \end{aligned}$$

This is also equal to 25,200 mi/hr, which is far faster than the initial velocity required to put a satellite in an orbit 100 mi high.

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**Exercise 12**

Calculate the escape velocity required to project a shell permanently away from the moon ($m_{moon} = 7.35\times10^{25}\ gm$, $r_{moon} = 1.74\times10^8\ cm$).

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**Exercise 13**

Once a shell has escaped from the earth, what must its speed be to allow it to escape from our solar system?

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**Exercise 14**

Find the escape velocities from the earth and the moon, using the planetary units given on page 8-14.

## BLACK HOLES

A special feature of satellite motion we have just seen is that we can tell whether or not a satellite can escape simply by comparing kinetic energy with the gravitational potential energy. If the satellite's positive kinetic energy is greater in magnitude than the negative gravitational potential energy, then the satellite escapes, never to return on its own. This is true no matter how or from where the satellite is launched (provided it does not crash into something.)

So far we have limited our discussion to slowly moving objects where the approximate formula $1/2\,mv^2$ is adequate to describe kinetic energy. We got the formula $1/2\,mv^2$ back in Equation 7 by expanding $E = mc^2$ to get

$$E = mc^2 = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}}$$

$$\approx \underset{\text{(rest energy)}}{m_0c^2} + \underset{\text{(kinetic energy)}}{1/2\,m_0v^2} \qquad (7)$$

The basic idea is that Einstein's formula $E = mc^2$ gives us a precise formula for the sum of the rest energy and the kinetic energy. In the special case the particle is moving slowly, we can use the approximate formula for $\sqrt{1 - v^2/c^2}$ to get the result shown in Equation 7.

For familiar objects like bullets, cars, airplanes, and rockets, the kinetic energy is $1/2\ m_0v^2$, much, much, much smaller than the rest energy $m_0c^2$. The kinetic energy of a rifle bullet, for example, is enough to allow the bullet to penetrate a few centimeters into a block of wood. The rest energy of the bullet, if converted into explosive energy, could destroy a forest. In fact, one way to tell whether or not the approximate formula $1/2\ m_0v^2$ is reasonably accurate, is to check whether the kinetic energy is much less than the rest energy. If it is, you can use the approximate formula; if not, you can't.

We now finish our discussion of satellite motion by going to the opposite extreme, and consider the behavior of particles whose kinetic energy is much greater than their rest energy. Such a particle must be moving at a speed very close to the speed of light. We considered such a particle in Exercise 7 of Chapter 6. There we saw that electrons emerging from the Stanford linear accelerator travelled at a speed $v = .9999999999875\ c$, and had a mass 200,000 times greater than the rest mass. For such a particle, almost all the energy is kinetic energy. In the formula $E = mc^2$, only one part in 200,000 represents rest energy.

Actually we wish to go one step farther, and discuss particles with no rest energy, particles that move ***at*** the speed of light. The obvious example, of course, is the photon, the particle of light itself.

From one point of view there is not much difference between an electron travelling at a speed $.9999999999875\ c$ with only 1 part in 200,000 of its energy in the form of rest energy, and a photon traveling at a speed c and no rest energy. Taking this point of view, we will take as the formula for the energy of a photon $E = mc^2$, and assume that this is pure kinetic energy.

Applying the formula $E = mc^2$ to a photon implies that a photon has a mass $m_p = E_p/c^2$. We will now make the assumption that this mass $m_{photon}$ is gravitational mass, and that photons have a gravitational potential energy $-GMm_p/r$ like other objects. Our assumption, which is slightly in error, is that Newtonian gravity, which is a non relativistic theory, applies to particles moving near to or at the speed of light. It turns out that Einstein's relativistic theory of gravity gives almost the same answers, that we are seldom off by more than a factor of 2 in our predictions.

Suppose we have a photon a distance r from a star of mass $M_s$. If the photon has a mass $m_p$, then the formula for the total energy of the photon, its kinetic energy $m_pc^2$ plus its gravitational potential energy $-GM_sm_p/r$ is

$$E_{tot} = m_pc^2 - \frac{GM_sm_p}{r}$$

Since $m_p$ appears in both terms, we can factor it out (and also take out a factor of $c^2$) to get

$$E_{tot} = m_pc^2\left[1 - \frac{GM_s}{rc^2}\right] \qquad (56)$$

Equation 56 applies only when the photon is outside the star, i.e., when the distance r is greater than the radius R of the star.

In most cases, the gravitational potential energy is much less than the kinetic energy of a photon, and gravity has little effect on the motion of the photon. For example, if a photon were grazing the surface of the sun (if r in Equation 56) were equal to the sun's radius $R_{sun}$) we would have

$$E_{tot} = m_pc^2\left[1 - \frac{GM_s}{R_sc^2}\right] \qquad (57)$$

Putting in numbers $M_s = 1.99 \times 10^{33}\text{gm}$, $R_s = 6.96 \times 10^{10}\text{cm}$ we have

$$\frac{GM_s}{R_sc^2} = \frac{6.67 \times 10^{-8}\dfrac{\text{cm}^3}{\text{gm sec}^2} \times 1.99 \times 10^{33}\text{gm}}{6.96 \times 10^{10}\text{cm} \times \left(3 \times 10^{10}\right)^2\dfrac{\text{cm}^2}{\text{sec}^2}}$$

$$= .00000212$$

Thus

$$E_{tot} = m_pc^2\left[1 - .00000212\right] \qquad (58)$$

From Equation 58 we see that when a photon is as close as it can get to the surface of the sun, the gravitational potential energy contributes very little to the total energy of the photon, only 2 parts in a million.

However, suppose that the a star had *the same mass as the sun but a much, much smaller radius*. If it's radius $R_s$ were small enough, the factor $\left[1 - GM_s/R_sc^2\right]$ in Equation 58 would become negative, and a photon grazing the surface of this star would have a negative total energy. The photon could not escape from the star. ***No photons emerging from the surface of such a star could escape, and the star would cease to emit light.***

Let us see how small the sun would have to be in order that it could no longer radiate light. That would happen when the factor $\left[1 - GM_s/R_sc^2\right]$ is zero, when photons emerging from the surface of the sun have zero total energy. Putting in numbers we get

$$\frac{GM_s}{R_sc^2} = 1 \qquad (59)$$

$$R_s = \frac{GM_s}{c^2}$$

$$= \frac{6.67 \times 10^{-8}\dfrac{\text{cm}^3}{\text{cm sec}^2} \times 1.99 \times 10^{33}\text{cm}}{\left(3 \times 10^{10}\right)^2\text{cm}^2/\text{sec}^2}$$

$$R_s = 1.48 \times 10^5\text{cm}$$

$$= 1.48 \text{ kilometer} \qquad (60)$$

Equation 60 tells us that an object with as much mass as the sun, confined to a sphere of radius less than 1.48 kilometers, cannot radiate light. Although we used the non relativistic Newtonian gravity in this calculation, Einstein's relativistic theory of gravity makes the same prediction.

In discussions of black holes, one often sees a reference to the radius of the black hole. What is usually meant is the radius given by Equation 59, the radius at which light can no longer escape if a mass $M_s$ is contained within a sphere of radius $R_s$.

Do black holes exist? Can so much mass be concentrated in such a small sphere? The question has been difficult to answer because black holes are hard to observe since they do not emit light. They have to be detected indirectly, from the gravitational pull they exert on neighboring matter. In the sky there are many binary star systems, systems in which two stars orbit about each other. In some examples we have observed a bright star orbiting about an invisible companion. Careful analysis of the orbit of the bright star suggests that the invisible companion may be a black hole. There is recent evidence that gigantic black holes, with the mass of millions of suns, exists at the center of many galaxies, including our own.

That a black hole cannot radiate light is only one of the peculiar properties of these objects. When so much matter is concentrated in such a small volume of space, the gravitational force becomes so great that other forces cannot resist the crushing force of gravity, and as far as we know, the matter inside the black hole collapses to a point—a zero sized or very, very small sized object. At the present time we do not have a good theory for what happens to the matter inside a black hole. (We will have more to say about black holes in later chapters.)

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**Exercise 15**

Studies of the motion of the stars in our galaxy suggests that at the center of our galaxy is a large amount of mass concentrated in a very small volume. For this problem, assume that a mass of 100 million suns is concentrated in the small volume. If this massive object is in fact a black hole, what is the radius from which light can no longer escape?

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## A Practical System of Units

In the CGS system of units, where we measure distance in centimeters, mass in grams and time in seconds, the unit of force is the dyne ($1 \text{ dyne} = 1 \text{ gm cm/sec}^2$) and the unit of energy is the erg ($1 \text{ erg} = 1 \text{ gm cm}^2/\text{sec}^2$). We have found the CGS system quite convenient for analyzing strobe photographs with 1 cm grids. But when we begin to talk about forces and particularly energy, the CGS system is often rather inconvenient. A force of one dyne is more on the scale of the force exerted by a fly doing push-ups than the kind of forces we deal with in the lab. A baseball pitched by Roger Clemens has a kinetic energy of over a million ergs and a 100 watt light bulb uses ten million ergs of electrical energy per second. The dyne and particularly the erg are much too small a unit for most every day situations.

In the MKS system of units, where we measure distance in meters, mass in kilograms and time in seconds, the unit of force is the newton and energy the joule. The force required to lift a 1 kilogram mass is 9.8 newtons (mg), and the energy of a Roger Clemens' pitch is over 10 joules. When working with practical electrical phenomena, the use of the MKS system is the only sensible thing to do. The unit of power, the watt, is one joule of energy per second. Thus a 100 watt light bulb consumes 100 joules of electrical energy per second. Volts and amperes are both MKS units, the corresponding CGS units are statvolts and esu, which are almost never used.

Where CGS units are far superior is in working with the basic theory of atoms, as for the case of the Bohr theory discussed in Chapter 36. This is because the electric force law has a much simpler form in CGS units. What we will do in the text from this point on is to use MKS units almost exclusively until we get through the chapters in electrical theory and applications. Then we will go back to the CGS system in most of our discussions of atomic phenomena.

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