Chapter 32

Maxwell's Equations

In 1860 James Clerk Maxwell summarized the entire content of the theory of electricity and magnetism in a few short equations. In this chapter we will review these equations and investigate some of the predictions one can make when the entire theory is available.

What does a complete theory of electricity and magnetism involve? We have to fully specify the electric field $\vec{E}$*, the magnetic field* $\vec{B}$*, and describe what effect the fields have when they interact with matter.*

The interaction is described by the Lorentz force law

$$\vec{F} = q\vec{E} + q\,\vec{v}\times\vec{B} \qquad (28\text{-}18)$$

which tells us the force exerted on a charge q by the $\vec{E}$ *and* $\vec{B}$ *fields. As long as we stay away from the atomic world where quantum mechanics dominates, then the Lorentz force law combined with Newton's second law fully explains the behavior of charges in the presence of electric and magnetic fields, whatever the origin of the fields may be.*

To handle the electric and magnetic fields, recall our discussion in Chapter 30 (on two kinds of fields) where we saw that any vector field can be separated into two parts; a divergent part like the electric field of static charges, and a solenoidal part like the electric field in a betatron or inductor. To completely specify a vector field, we need two equations – one involving a surface integral or its equivalent to define the divergent part of the field, and another involving a line integral or its equivalent defining the solenoidal part.

In electricity theory we have two vector fields $\vec{E}$ *and* $\vec{B}$*, and two equations are needed to define each field. Therefore the total number of equations required must be four.*

How many of the required equations have we discussed so far? We have Gauss' law for the divergent part of $\vec{E}$*, and Faraday's law for the solenoidal part. It appears that we already have a complete theory of the electric field, and we do. Gauss' law and Faraday's law are two of the four equations needed.*

For magnetism, we have Ampere's law that defines the solenoidal part of $\vec{B}$*. But we have not written an equation involving the surface integral of* $\vec{B}$*. We are missing a Gauss' law type equation for the magnetic field. It would appear that the missing Gauss' law for* $\vec{B}$*, plus Ampere's law make up the remaining two equations.*

This is not quite correct. The missing Gauss' law is one of the needed equations for $\vec{B}$*, and it is easily written down because there are no known sources for a divergent* $\vec{B}$ *field. But Ampere's law, in the form we have been using*

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 i \qquad (29\text{-}18)$$

has a logical flaw that was discovered by Maxwell. When Maxwell corrected this flaw by adding another source term to the right side of Equation (29-18), he then had the complete, correct set of four equations for $\vec{E}$ *and* $\vec{B}$*.*

All Maxwell did was to add one term to the four equations for $\vec{E}$ and $\vec{B}$, and yet the entire set of equations are named after him. The reason for this is that with the correct set of equations, Maxwell was able to obtain solutions of the four equations, predictions of these equations that could not be obtained until Ampere's law had been corrected. The most famous of these predictions was that a certain structure of electric and magnetic fields could travel through empty space at a speed $v = 1/\sqrt{\mu_0\varepsilon_0}$. Since Maxwell knew that $1/\sqrt{\mu_0\varepsilon_0}$ was close to the observed speed $3 \times 10^8\ m/s$ for light, he proposed that this structure of electric and magnetic fields was light itself.

In this chapter, we will first describe the missing Gauss' law for magnetic fields, then correct Ampere's law to get the complete set of Maxwell's four equations. We will then solve these equations for a structure of electric and magnetic fields that moves through empty space at a speed $v = 1/\sqrt{\mu_0\varepsilon_0}$. We will see that this structure explains various properties of light waves, radio waves, and other components of the electromagnetic spectrum. We will find, for example, that we can detect radio waves by using the same equipment and procedures we have used in earlier chapters to detect and map electric and magnetic fields.

GAUSS' LAW FOR MAGNETIC FIELDS

Let us review a calculation we have done several times now—the use of Gauss' law to calculate the electric field of a point particle. Our latest form of the law is

$$\int_{\substack{\text{closed}\\\text{surface}}} \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\varepsilon_0} \tag{29-5}$$

where Q_{in} is the total amount of electric charge inside the surface.

In Figure (1), we have a point charge Q and have constructed a closed spherical surface of radius r centered on the charge. For this surface, $\vec{E}$ is everywhere perpendicular to the surface or parallel to every surface element $d\vec{A}$, thus $\vec{E} \cdot d\vec{A} = E\,dA$. Since E is of constant magnitude, we get

$$\begin{aligned}\int_{\substack{\text{closed}\\\text{surface}}} \vec{E} \cdot d\vec{A} &= E\int_{\substack{\text{closed}\\\text{surface}}} dA\\ &= E\,4\pi r^2\\ &= \frac{Q_{in}}{\varepsilon_0}\end{aligned} \tag{1}$$

where $Q_{in} = Q$.

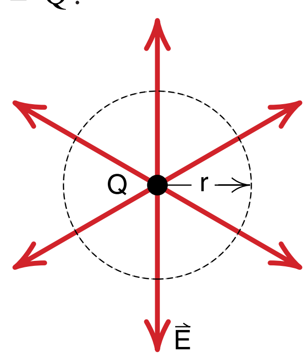


Figure 1
Field of point charge.

The solution of Equation (1) gives $E = Q/4\pi\varepsilon_0 r^2$ as the strength of the electric field of a point charge. A similar calculation using a cylindrical surface gave us the electric field of a charged rod. By being clever, or working very hard, one can use Gauss' law in the form of Equation (29-5) to solve for the electric field of any static distribution of electric charge.

But the simple example of the field of a point charge illustrates the point we wish to make. Gauss' law determines the diverging kind of field we get from a point source. Electric fields have point sources, namely electric charges, and it is these sources in the form of Q_{in} that appear on the right hand side of Equation (29-5).

Figure (2) shows a magnetic field emerging from a point source of magnetism. Such a point source of magnetism is given the name ***magnetic monopole*** and magnetic monopoles are predicted to exist by various recent theories of elementary particles. These theories are designed to unify three of the four basic interactions – the electrical, the weak, and the nuclear interactions. (They are called "Grand Unified Theories" or "GUT" theories. Gravity raises problems that are not handled by GUT theories.) These theories also predict that the proton should decay with a half life of 10^{32} years.

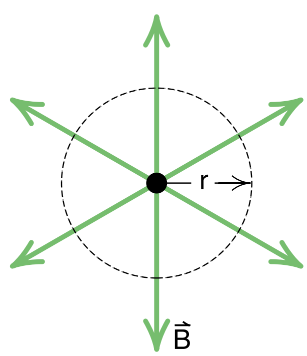


Figure 2
Magnetic field produced by a point source.

In the last 20 years there has been an extensive search for evidence for the decay of protons or the existence of magnetic monopoles. So far we have found no evidence for either. (You do not have to wait 10^{32} years to see if protons decay; instead you can see if one out of 10^{32} protons decays in one year.)

The failure to find the magnetic monopole, the fact that no one has yet seen a magnetic field with the shape shown in Figure (2), can be stated mathematically by writing a form of Gauss' law for magnetic fields with the magnetic charge Q_{in} set to zero

$$\int_{\substack{\text{closed}\\\text{surface}}} \vec{B}\cdot d\vec{A} = 0 \tag{2}$$

When reading Equation (2), interpret the zero on the right side of Equation (2) as a statement that the divergent part of the magnetic field has no ***source term***. This is in contrast to Gauss' law for electric fields, where Q_{in}/ε_0 is the source term.

MAXWELL'S CORRECTION TO AMPERE'S LAW

As we mentioned in the introduction, Maxwell detected a logical flaw in Ampere's law which, when corrected, gave him the complete set of equations for the electric and magnetic fields. With the complete set of equations, Maxwell was able to obtain a theory of light. No theory of light could be obtained without the correction.

Ampere's law, Equation (29-18), uses the line integral to detect the solenoidal component of the magnetic field. We had

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 i_{enclosed} \qquad \textit{Ampere's Law} \qquad (29\text{-}18)$$

where $i_{enclosed}$ is the total current encircled by the closed path used to evaluate $\vec{B} \cdot d\vec{\ell}$. We can say that $\mu_0 i_{enclosed}$ is the source term for this equation, in analogy to Q_{in}/ε_0 being the source term for Gauss' law.

Before we discuss Maxwell's correction, let us review the use of Equation (29-18) to calculate the magnetic field of a straight current i as shown in Figure (3). In (3a) we see the wire carrying the current, and in (3b) we show the circular magnetic field produced by the current. To apply Equation (29-18) we draw a closed circular path of radius r around the wire, centering the path on the wire so that $\vec{B}$ and sections $d\vec{\ell}$ of the path are everywhere parallel. Thus $\vec{B} \cdot d\vec{\ell} = B\,d\ell$, and since B is constant along the path, we have

$$\oint \vec{B} \cdot d\vec{\ell} = B\oint d\ell = B \times 2\pi r = \mu_0 i$$

which gives our old formula $B = \mu_0 i/2\pi r$ for the magnetic field of a wire.

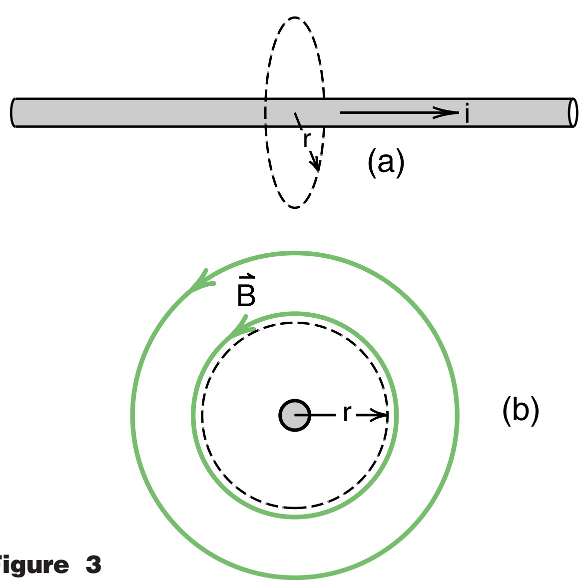


Figure 3
Using Ampere's law.

To see the flaw with Ampere's law, consider a circuit where a capacitor is being charged up by a current i as shown in Figure (4). When a capacitor becomes charged, one plate becomes positively charged and the other negatively charged as shown. We can think of the capacitor being charged because a positive current is flowing into the left plate, making that plate positive, and a positive current is flowing out of the right plate, making that plate negative.

Figure (4) looks somewhat peculiar in that the current i almost appears to be flowing through the capacitor. We have a current i on the left, which continues on the right, with a break between the capacitor plates. To emphasize the peculiar nature of this discontinuity in the current, imagine that the wires leading to the capacitor are huge wires, and that the capacitor plates are just the ends of the wires as shown in Figure (5).

Now let us apply Ampere's law to the situation shown in Figure (5). We have drawn three paths, Path (1) around the wire leading into the positive plate of the capacitor, Path (2) around the wire leading out of the negative plate, and Path (3) around the gap between the plates. Applying Ampere's law we have

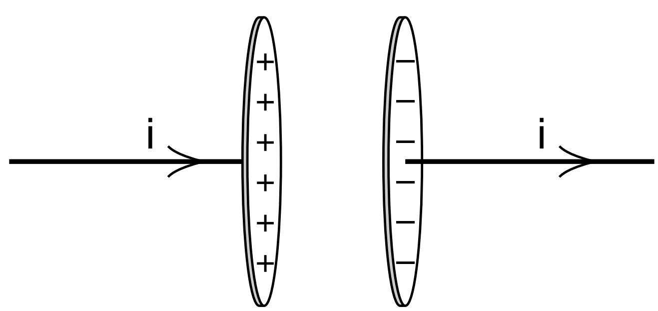


Figure 4
Charging up a capacitor.

$$\oint_{\text{path 1}} \vec{B}\cdot d\vec{\ell} = \mu_0 i \qquad \textit{path 1 goes around a current } i \qquad (3)$$

$$\oint_{\text{path 2}} \vec{B}\cdot d\vec{\ell} = \mu_0 i \qquad \textit{path 2 goes around a current } i \qquad (4)$$

$$\oint_{\text{path 3}} \vec{B}\cdot d\vec{\ell} = 0 \qquad \textit{path 3 does not go around any current} \qquad (5)$$

When we write out Ampere's law this way, the discontinuity in the current at the capacitor plates looks a bit more disturbing.

For greater emphasis of the problem, imagine that the gap in Figure (5) is very narrow, like Figure (5a) only worse. Assume we have a 1 mm diameter wire and the gap is only 10 atomic diameters. Then according to Ampere's law, $\oint \vec{B}\cdot d\vec{\ell}$ should still be zero if it is correctly centered on the gap. But can we possibly center a path on a gap that is only 10 atomic diameters wide? And even if we could, would $\oint \vec{B}\cdot d\vec{\ell}$ be zero for this path, and have the full value $\mu_0 i$ for the path 10 atomic diameters away? No, we simply cannot have such a discontinuity in the magnetic field and there must be something wrong with Ampere's law. This was the problem recognized by Maxwell.

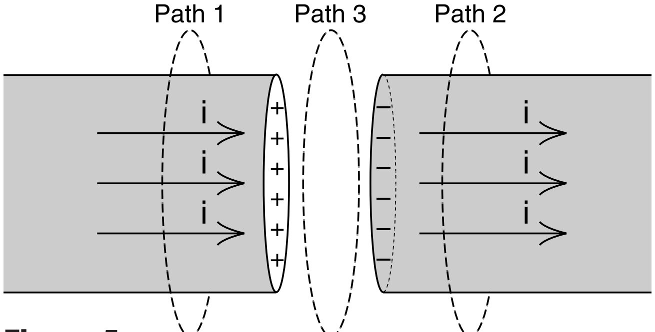


Figure 5
Current flows through Paths (1) & (2), but not through Path (3).

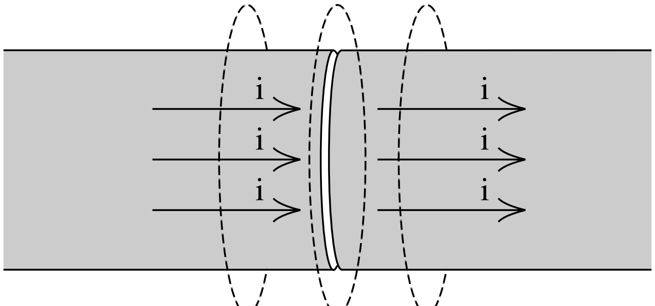


Figure 5a
Very narrow gap

Maxwell's solution was that even inside the gap at the capacitor there was a source for $\oint \vec{B}\cdot d\vec{\ell}$, and that the strength of the source was still $\mu_0 i$. What actually exists inside the gap is the electric field $\vec{E}$ due to the + and – charge accumulating on the capacitor plates as shown in Figure (6). Perhaps this electric field can somehow replace the missing current in the gap.

The capacitor plates or rod ends in Figure (6) have a charge density $\sigma = Q/A$ where Q is the present charge on the capacitor and A is the area of the plates. In one of our early Gauss' law calculations we saw that a charge density on a conducting surface produces an electric field of strength $E = \sigma/\varepsilon_0$, thus E between the plates is related to the charge Q on them by

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A} \quad ; \quad Q = \varepsilon_0 EA \qquad (6)$$

The current flowing into the capacitor plates is related to the charge Q that has accumulated by

$$i = \frac{dQ}{dt} \qquad (7)$$

Using Equation (6) in Equation (7), we get

$$i = \varepsilon_0 \frac{d}{dt}(EA) \qquad (8)$$

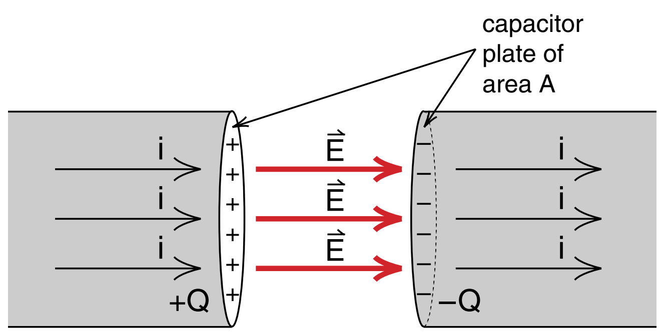


Figure 6
An electric field $\vec{E}$ exists between the plates.

Noting that the flux Φ_E of electric field between the plates is

$$\Phi_E = \vec{E}\cdot d\vec{A} = E A_\perp = E A \qquad (9)$$

and multiplying through by μ_0, we can write Equation (8) in the form

$$\boxed{\mu_0 i = \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}} \qquad (10)$$

We get the somewhat surprising result that $\mu_0\varepsilon_0$ times the rate of change of electric flux inside the capacitor has the same magnitude as $\mu_0 i$, where i is the current in the wire leading to the capacitor. Maxwell proposed that $\mu_0\varepsilon_0 d\Phi_E/dt$ played the same role, inside the capacitor, as a source term for $\oint \vec{B}\cdot d\vec{l}$, that $\mu_0 i$ did outside in the wire.

As a result, Maxwell proposed that Ampere's law be corrected to read

$$\boxed{\oint \vec{B}\cdot d\vec{\ell} = \mu_0 i + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}} \qquad \textit{corrected Ampere's law} \qquad (11)$$

Applying Equation (11) to the three paths shown in Figure (7), we have

$$\oint_{\text{paths 1\&2}} \vec{B}\cdot d\vec{\ell} = \mu_0 i \qquad (\Phi_E = 0) \qquad (12)$$

$$\oint_{\text{paths 3}} \vec{B}\cdot d\vec{\ell} = \mu_0\varepsilon_0 \frac{d\Phi_E}{dt} \qquad (i = 0) \qquad (13)$$

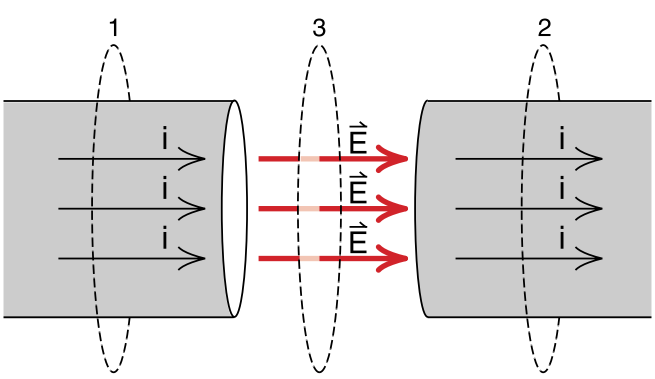


Figure 7
Path (2) surrounds a changing electric flux. Inside the gap, $\mu_0\varepsilon_0(d\Phi_E/dt)$ replaces $\mu_0 i$ as the source of $\vec{B}$.

I.e., for Paths (1) and (2), there is no electric flux through the path and the source of $\oint \vec{B}\cdot d\vec{\ell}$ is the current. For Path (3), no current flows through the path and the source of $\oint \vec{B}\cdot d\vec{\ell}$ is the changing electric flux. But, because $\mu_0 i$ in Equation (12) has the same magnitude $\mu_0\varepsilon_0 d\Phi_E/dt$ in Equation (13), the term $\oint \vec{B}\cdot d\vec{\ell}$ has the same value for Path (3) as (1) and (2), and there is no discontinuity in the magnetic field.

Example: Magnetic Field between the Capacitor Plates

As an example of the use of the new term in the corrected Ampere's law, let us calculate the magnetic field in the region between the capacitor plates. To do this we draw a centered circular path of radius r smaller than the capacitor radius R as shown in Figure (8). There is no current through this path, but there is an electric flux $\Phi_E(r) = EA(r) = E\pi r^2$ through the path. Thus we set $i = 0$ in Ampere's corrected law, and replace Φ_E by the flux $\Phi_E(r)$ through our path to get

$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0\varepsilon_0 \frac{d\Phi_E(r)}{dt} \qquad (14)$$

Equation (14) tells us that because we have an increasing electric field between the plates, and thus an increasing electric flux through our path, there must be a magnetic field around the path.

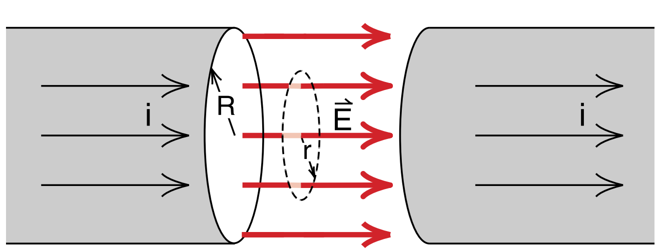


Figure 8
Calculating $\vec{B}$ in the region between the plates.

Due to the cylindrical symmetry of the problem, the only possible shape for the magnetic field inside the capacitor is circular, just like the field outside. This circular field and our path are shown in the end view, Figure (9). Since $\vec{B}$ and $d\vec{\ell}$ are parallel for all the steps around the circular path, we have $\vec{B} \cdot d\vec{\ell} = B\,d\ell$. And since B is constant in magnitude along the path, we get

$$\oint \vec{B} \cdot d\vec{\ell} = B\oint d\ell = B \times 2\pi r \qquad (15)$$

To evaluate the right hand side of Equation (14), note that the flux through our path $\Phi_E(r)$ is equal to the total flux $\Phi_E(\text{total})$ times the ratio of the area πr^2 of our path to the total area πR^2 of the capacitor plates

$$\Phi_E(r) = \Phi_E(\text{total})\frac{\pi r^2}{\pi R^2} \qquad (16)$$

so that the right hand side becomes

$$\mu_0\varepsilon_0\frac{d\Phi_E(r)}{dt} = \mu_0\varepsilon_0\frac{d\Phi_E(\text{total})}{dt}\frac{r^2}{R^2}$$

$$= \mu_0 i\frac{r^2}{R^2} \qquad (17)$$

where in the last step we used Equation (10) to replace $\mu_0\varepsilon_0 d\Phi_E(\text{total})/dt$ by a term of the same magnitude, namely $\mu_0 i$.

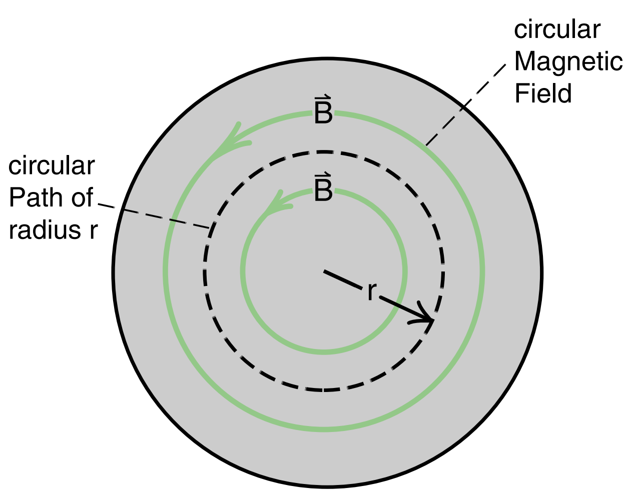


Figure 9
End view of capacitor plate.

Finally using Equation (15) and (17) in (14) we get

$$B \times 2\pi r = \mu_0 i\frac{r^2}{R^2}$$

$$B = \frac{\mu_0 i}{2\pi}\left(\frac{r}{R^2}\right) \qquad \textit{magnetic field between capacitor plates} \qquad (18)$$

Figure (10) is a graph of the magnitude of B both inside and outside the plates. They match up at r = R, and the field strength decreases linearly to zero inside the plates.

Exercise 1

Calculate the magnetic field inside the copper wires that lead to the capacitor plates of Figure (5). Use Ampere's law and a circular path of radius r inside the copper as shown in Figure (11). Assuming that there is a uniform current density in the wire, you should get Equation (18) as an answer. Thus the magnetic field is continuous as we go out from the copper to between the capacitor plates.

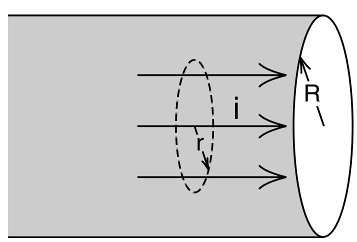


Figure 11

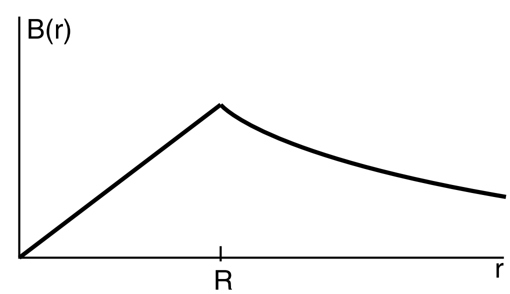


Figure 10
Magnetic fields inside and outside the gap.

MAXWELL'S EQUATIONS

Now that we have corrected Ampere's law, we are ready to write the four equations that completely govern the behavior of classical electric and magnetic fields. They are

$$\text{(a)} \quad \int_{\substack{\text{closed}\\ \text{surface}}} \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\varepsilon_0} \qquad \textit{Gauss' Law}$$

$$\text{(b)} \quad \int_{\substack{\text{closed}\\ \text{surface}}} \vec{B} \cdot d\vec{A} = 0 \qquad \textit{No Monopole}$$

$$\text{(c)} \quad \oint \vec{B} \cdot d\vec{\ell} = \mu_0 i + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \qquad \textit{Ampere's Law}$$

$$\text{(d)} \quad \oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt} \qquad \textit{Faraday's Law}$$

(19)

The only other thing you need for the classical theory of electromagnetism is the Lorentz force law and Newton's second law to calculate the effect of electric and magnetic fields on charged particles.

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \qquad \textit{Lorentz Force Law}$$

(20)

This is a complete formal summary of everything we have learned in the past ten chapters.

Exercise 2

This is one of the most important exercises in the text. The four Maxwell's equations and the Lorentz force law represent an elegant summary of many ideas. But these equations are nothing but hen scratchings on a piece of paper if you do not have a clear idea of how each term is used.

The best way to give these equations meaning is to know inside out at least one specific example that illustrates the use of each term in the equations. For Gauss' law, we have emphasized the calculation of the electric field of a point and a line charge. We have the nonexistence of the divergent magnetic field in Figure (2) to illustrate Gauss' law for magnetic fields. We have used Ampere's law to calculate the magnetic field of a wire and a solenoid. The new term in Ampere's law was used to calculate the magnetic field inside a parallel plate capacitor that is being charged up.

Faraday's law has numerous applications including the air cart speed meter, the betatron, the AC voltage generator, and the inductance of a solenoid. Perhaps the most important concept with Faraday's law is that $\oint \vec{E} \cdot d\vec{l}$ is the voltage rise created by solenoidal electric fields, which for circuits can be read directly by a voltmeter. This lead to the interpretation of a loop of wire with a voltmeter attached as an $\oint \vec{E} \cdot d\vec{l}$ meter. We used an $\oint \vec{E} \cdot d\vec{l}$ meter in the design of the air cart speed detector and experiment where we mapped the magnetic field of a Helmholtz coil.

Then there is the Lorentz force law with the formulas for the electric and magnetic force on a charged particle. As an example of an electric force we calculated the trajectory of an electron beam between charged plates, and for a magnetic force we studied the circular motion of electrons in a uniform magnetic field.

The assignment of this exercise is to write out Maxwell's equations one by one, and with each equation write down a fully worked out example of the use of each term. Do this neatly, and save it for later reference. This is what turns the hen scratchings shown on the previous page into a meaningful theory. When you buy a T-shirt with Maxwell's equations on it, you will be able to wear it with confidence.

We have just crossed what you might call a continental divide in our study of the theory of electricity and magnetism. We spent the last ten chapters building up to Maxwell's equations. Now we descend into applications of the theory. We will focus on applications and discussions that would not have made sense until we had the complete set of equations—discussions on the symmetry of the equations and applications like Maxwell's theory of light.

SYMMETRY OF MAXWELL'S EQUATIONS

Maxwell's Equations (19 a, b, c, and d), display considerable symmetry, and a special lack of symmetry. But the symmetry or lack of it is clouded by our choice of the MKS units with its historical constants μ_0 and ε_0 that appear, somewhat randomly, either in the numerator or denominator at various places.

For this section, let us use a special set of units where the constants μ_0 and ε_0 have the value 1

$$\mu_0 = 1 ; \quad \varepsilon_0 = 1 \quad \left(\begin{array}{c}\text{in a special}\\ \text{set of units}\end{array}\right) \tag{21}$$

Because the speed of light c is related to μ_0 and ε_0 by $c = 1/\sqrt{\mu_0\varepsilon_0}$, we are now using a set of units where the speed of light is 1. If we set $\mu_0 = \varepsilon_0 = 1$ in Equations (19) we get

$$\int_{\substack{\text{closed}\\ \text{surface}}} \vec{E}\cdot d\vec{A} = Q_{in} \tag{22a}$$

$$\int_{\substack{\text{closed}\\ \text{surface}}} \vec{B}\cdot d\vec{A} = 0 \tag{22b}$$

$$\oint \vec{B}\cdot d\vec{\ell} = i + \frac{d\Phi_E}{dt} \tag{22c}$$

$$\oint \vec{E}\cdot d\vec{\ell} = -\frac{d\Phi_B}{dt} \tag{22d}$$

Stripping out μ_0 and ε_0 gives a clearer picture of what Maxwell's equations are trying to say. Equation (22a) tells us that electric charge is the source of divergent electric fields. Equation (22b) says that we haven't found any source for divergent magnetic fields. Equation (22c) tells us that an electric current or a changing electric flux is a source for solenoidal magnetic fields, and (22d) tells us that a changing magnetic flux creates a solenoidal electric field.

Equations (22) immediately demonstrate the lack of symmetry caused by the absence of magnetic monopoles, and so does the Lorentz force law of Equation (20). If the magnetic monopole is discovered, and we assign to it the "magnetic charge" Q_B, then for example Equation (22b) would become

$$\int_{\substack{\text{closed}\\ \text{surface}}} \vec{B}\cdot d\vec{A} = Q_B \tag{22b'}$$

If we have magnetic monopoles, a magnetic field should exert a force $\vec{F}_B = Q_B\vec{B}$ and perhaps an electric field should exert a force something like $\vec{F}_E = Q_B\vec{v}\times\vec{E}$.

Aside from Equation (22b), the other glaring asymmetry is the presence of an electric current i in Ampere's law (22c) but no current term in Faraday's law (22d). If, however, we have magnetic monopoles we can also have a current i_B of magnetic monopoles, and this asymmetry can be removed.

Exercise 3

Assume that the magnetic monopole has been discovered, and that we now have magnetic charge Q_B and a current i_B of magnetic charge. Correct Maxwell's Equations (22) and the Lorentz force law (20) to include the magnetic monopole. For each new term you add to these equations, provide a worked-out example of its use.

In this exercise, use symmetry to guess what terms should be added. If you want to go beyond what we are asking for in this exercise, you can start with the formula $\vec{F}_B = Q_B\vec{B}$ for the magnetic force on a magnetic charge, and with the kind of thought experiments we used in the chapter on magnetism, derive the formula for the electric force on a magnetic charge Q_B. You will also end up with a derivation of the correction to Faraday's law caused by a current of magnetic charge. (This is more of a project than an exercise.)

MAXWELL'S EQUATIONS IN EMPTY SPACE

In the remainder of this chapter we will discuss the behavior of electric and magnetic fields in empty space where there are no charges or currents. A few chapters ago, there would not have been much point in such a discussion, for electric fields were produced by charges, magnetic fields by currents, and without charges and currents, we had no fields. But with Faraday's law, we see that a changing magnetic flux $d\Phi_B/dt$ acts as the source of a solenoidal electric field. And with the correction to Ampere's law, we see that a changing electric flux is a source of solenoidal magnetic fields. Even without charges and currents we have sources for both electric and magnetic fields.

First note that if we have no electric charge (or magnetic monopoles), then we have no sources for either a divergent electric or divergent magnetic field. In empty space diverging fields do not play an important role and we can focus our attention on the equations for the solenoidal magnetic and solenoidal electric field, namely Ampere's and Faraday's laws.

Setting $i = 0$ in Equation (19c), the Equations (19c) and (19d) for the solenoidal fields in empty space become

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \tag{23a}$$

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt} \tag{23b}$$

We can make these equations look better if we write $\mu_0\varepsilon_0$ as $1/c^2$, where $c = 3 \times 10^8$ m/s as determined in our LC circuit experiment. Then Equations (23) become

$$\oint \vec{B} \cdot d\vec{\ell} = \frac{1}{c^2}\frac{d\Phi_E}{dt} \tag{24a}$$

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt} \tag{24b}$$

Maxwell's equations in empty space

Equations (24a, b) suggest a coupling between electric and magnetic fields. Let us first discuss this coupling in a qualitative, somewhat sloppy way, and then work out explicit examples to see precisely what is happening.

Roughly speaking, Equation (24a) tells us that a changing electric flux or field creates a magnetic field, and (24b) tells us that a changing magnetic field creates an electric field. These fields interact, and in some sense support each other.

If we were experts in integral and differential equations, we would look at Equations (24) and say, "Oh, yes, this is just one form of the standard wave equation. The solution is a wave of electric and magnetic fields traveling through space." Maxwell was able to do this, and solve Equations (24) for both the structure and the speed of the wave. The speed turns out to be c, and he guessed that the wave was light.

Because the reader is not expected to be an expert in integral and differential equations, we will go slower, working out specific examples to see what kind of structures and behavior we do get from Equations (24). We are just beginning to touch upon the enormous subject of electromagnetic radiation.

A Radiated Electromagnetic Pulse

We will solve Equations (24) the same way we have been solving all equations involving derivatives or integrals—by guessing and checking. The rules of the game are as follows. Guess a solution, then apply Equations (24) to your guess in every possible way you can think of. If you cannot find an inconsistency, your guess may be correct.

In order to guess a solution, we want to pick an example that we know as much as possible about and use every insight we can to improve our chances of getting the right answer. Since we are already familiar with the fields associated with a current in a wire, we will focus on that situation. Explicitly, we will consider what happens, what kind of fields we get, when we first turn on a current in a wire. We will see that a structure of magnetic and electric fields travels out from the wire, in what will be an example of a radiated electromagnetic pulse or wave.

A Thought Experiment

Let us picture a very long, straight, copper wire with no current in it. At time $t = 0$ we start an upward directed current i everywhere in the wire as shown in Figure (12). This is the tricky part of the experiment, having the current i start everywhere at the same time. If we closed a switch, the motion of charge would begin at the switch and advance down the wire. To avoid this, imagine that we have many observers with synchronized watches, and they all reach into the wire and start the positive charge moving at $t = 0$. However you want to picture it, just make sure that there is no current in the wire before $t = 0$, and that we have a uniform current i afterward.

In our previous discussions, we saw that a current i in a straight wire produced a circular magnetic field of magnitude $B = \mu_0 i/2\pi r$ everywhere outside the wire. This cannot be the solution we need because it implies that as soon as the current is turned on, we have a magnetic field throughout all of space. The existence of the magnetic field carries the information that we have turned on the current. Thus the instantaneous spread of the field throughout space carries this information faster than the speed of light and violates the principle of causality. As we saw in Chapter 1, we could get answers to questions that have not yet been asked.

Using our knowledge of special relativity as a guide, we suspect that the solution $B = \mu_0 i/2\pi r$ everywhere in space, instantaneously, is not a good guess. A more reasonable guess is that the magnetic field grows at some speed v out from the wire. Inside the growing front, the field may be somewhat like its final form $B = \mu_0 i/2\pi r$, but outside we will assume $B = 0$.

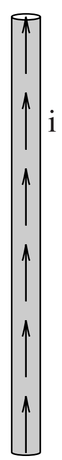


Figure 12
A current i is started all along the wire at time t = 0.

The pure, expanding magnetic field shown in Figure (13) seems like a good guess. But it is wrong, as we can see if we apply Ampere's law to Path (a) which has not yet been reached by the growing magnetic field. For this path that lies outside the magnetic field, $\oint \vec{B} \cdot d\vec{\ell} = 0$, and the corrected Ampere's law, Equation (19c), gives

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 i + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = 0 \qquad (25)$$

In our picture of Figure (13) we have no electric field, therefore $\Phi_E = 0$ and Equation (25) implies that $\mu_0 i$ is zero, or the current i through Path (a) is zero. But **the current is not zero** and we thus have an inconsistency. The growing magnetic field of Figure (13) is not a solution of Maxwell's equations. (This is how we play the game. Guess and try, and this time we failed.)

Equation (25) gives us a hint of what is wrong with our guess. It says that

$$\frac{d\Phi_E}{dt} = -\frac{i}{\varepsilon_0} \qquad (25a)$$

thus if we have a current i and have the growing magnetic field shown in Figure (13) we must also have a changing electric flux Φ_E through Path (a). Somewhere there must be an electric field $\vec{E}$ to produce the changing flux Φ_E, a field that points either up or down, passing through the circular path of Figure (13).

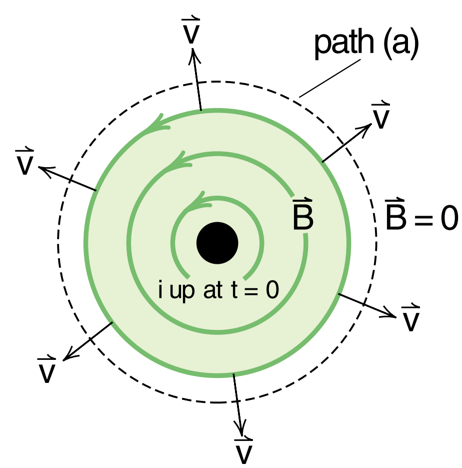


Figure 13
As a guess, we will assume that the magnetic field expands at a speed v out from the wire, when the current is turned on.

In our earlier discussion of inductance and induced voltage, we saw that a changing current creates an electric field that opposed the change. This is what gives an effective inertia to the current in an inductor. Thus when we suddenly turn on the upward directed current as shown in Figure (12), we expect that we should have a downward directed electric field as indicated in Figure (14), opposing our trying to start the current.

Initially the downward directed electric field should be inside the wire where it can act on the current carrying charges. But our growing circular magnetic field shown in Figure (13) must also have started inside the wire. Since a growing magnetic field alone is not a solution of Maxwell's equations and since there must be an associated electric field, let us propose that both the circular magnetic field of Figure (13), and the downward electric field of Figure (14) grow together as shown in Figure (15).

In Figure (15), we have sketched a field structure consisting of a circular magnetic field and a downward electric field that started out at the wire and is expanding radially outward at a speed v as shown. This structure has not yet expanded out to our Path (1), so that the line integral $\oint \vec{B}\cdot d\vec{\ell}$ is still zero and Ampere's law still requires that

$$0 = \mu_0 i + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt} \qquad \textit{Path 1 of Figure 15}$$

$$\frac{d\Phi_E}{dt} = -\frac{i}{\varepsilon_0} \tag{26}$$

which is the same as Equation (25).

Looking at Figure (15), we see that the downward electric field gives us a negative flux Φ_E through our path. (We chose the direction of the path so that by the right hand convention, the current i is positive.) And as the field structure expands, we have more negative flux through the path. This increasing negative flux is just what is required by Equation (26).

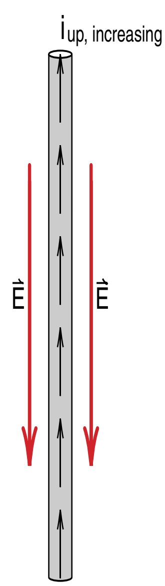


Figure 14
When a current starts up, it is opposed by an electric field.

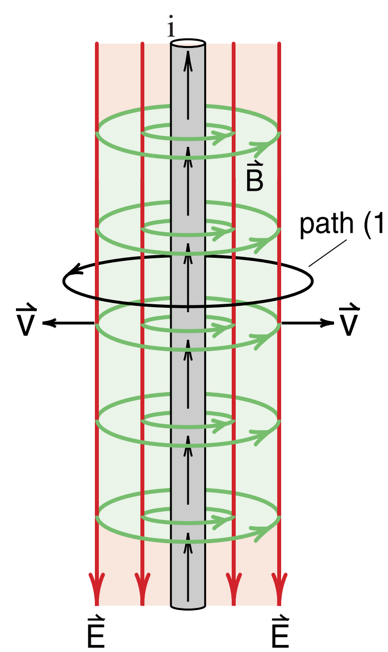


Figure 15
As a second guess, we will assume that there is a downward directed electric field associated with the expanding magnetic field. Again, Path (1) is out where the fields have not yet arrived.

What happens when the field structure gets to and passes our path? The situation suddenly changes. Now we have a magnetic field at the path, so that $\oint \vec{B} \cdot d\vec{\ell}$ is no longer zero. And now the expanding front is outside our path so that the expansion no longer contributes to $-d\Phi_E/dt$. The sudden appearance of $\oint \vec{B} \cdot d\vec{\ell}$ is precisely compensated by the sudden loss of the $d\Phi_E/dt$ due to expansion of the field structure.

The alert student, who calculates $\oint \vec{E} \cdot d\vec{\ell}$ for some paths inside the field structure of Figure (15) will discover that we have not yet found a completely satisfactory solution to Maxwell's equations. The electric fields in close to the wire eventually die away, and only when they have gone do we get a static magnetic field given by $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 i + 0$.

The problems associated with the electric field dying away can be avoided if we turn on the current at time $t = 0$, and then shut it off a very short time later. In that case we should expect to see an expanding cylindrical shell of electric and magnetic fields as shown in Figure (16). The front of the shell started out when the current was turned on, and the back should start out when the current is shut off. We will guess that the front and back should both travel radially outward at a speed v as shown.

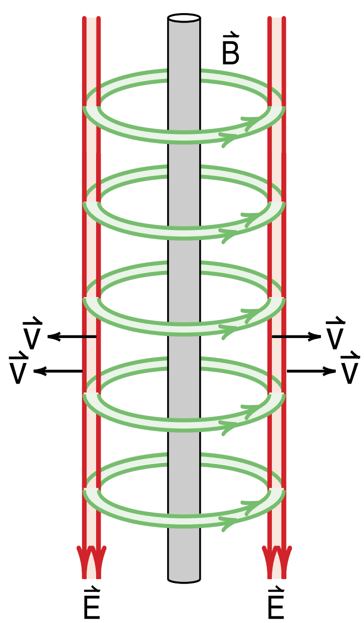


Figure 16
Electromagnetic pulse produced by turning the current on and then quickly off. We will see that this structure agrees with Maxwell's equations.

Speed of an Electromagnetic Pulse

Let us use Figure (16), redrawn as Figure (17a), as our best guess for the structure of an electromagnetic pulse. The first step is to check that this field structure obeys Maxwell's equations. If it does, then we will see if we can solve for the speed v of the wave front.

In Figure (17a), where we have shut the current off, there is no net charge or current and all we need to consider is the expanding shell of electric and magnetic fields moving through space. We have no divergent fields, no current, and the equations for $\vec{E}$ and $\vec{B}$ become

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \qquad (23a)$$

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt} \qquad (23b)$$

which we wrote down earlier as Maxwell's equation for empty space.

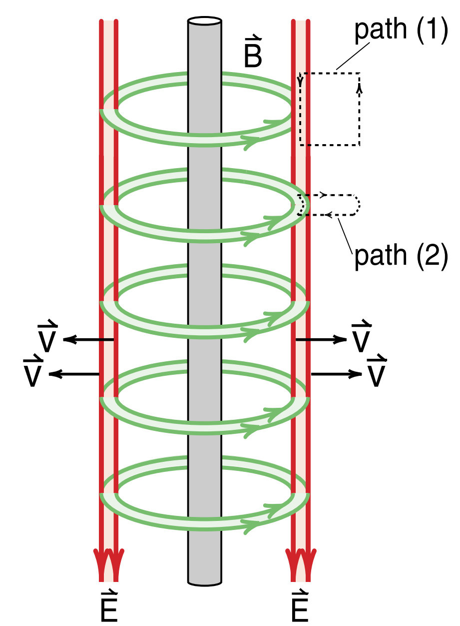


Figure 17a
In order to analyze the electromagnetic pulse produced by turning the current on and off, we introduce the two paths shown above. Path (1) has one side parallel to the electric field, while Path (2) has a side parallel to the magnetic field.

In order to apply Maxwell's equations to the fields in Figure (17a), we will focus our attention on a small piece of the shell on the right side that is moving to the right at a speed v. For this analysis, we will use the two paths labeled Path (1) and Path (2). Path (1) has a side parallel to the electric field, and will be used for Equation 23b. Path (2) has a side parallel to the magnetic field, and will be used for Equation 23a.

Analysis of Path 1

In Figure (17b), we have a close up view of Path (1). The path was chosen so that only the left edge of length h was in the electric field, so that

$$\oint \vec{E} \cdot d\vec{\ell} = Eh \qquad (27)$$

In order to make $\vec{E} \cdot d\vec{\ell}$ positive on this left edge, we went around Path (1) in a counterclockwise direction. By the right hand convention, any vector up through this path is positive, therefore the downward directed magnetic field is going through Path (1) in a negative direction. (We will be very careful about signs in this discussion.)

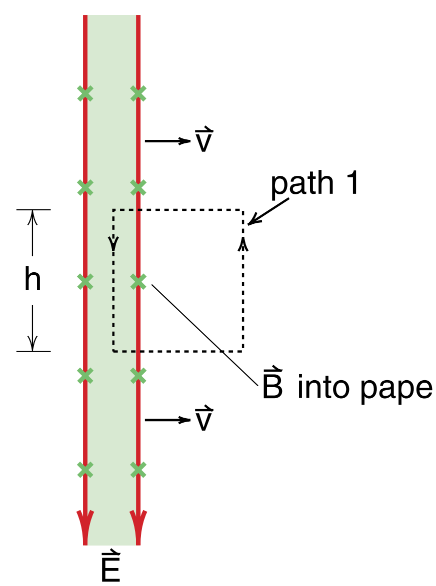


Figure 17b
Side view showing path (1). An increasing (negative) magnetic flux flows down through Path (1).

In Figure (18), we are looking at Path (1), first at a time t (18a) where the expanding front has reached a position x as shown, then at a time $t+\Delta t$ where the front has reached $x + \Delta x$. Since the front is moving at an assumed speed v , we have

$$v = \frac{\Delta x}{\Delta t}$$

At time $t+\Delta t$, there is additional magnetic flux through Path (1). The amount of additional magnetic flux $\Delta\Phi_B$ is equal to the strength B of the field times the additional area ($h\Delta x$). Since $\vec{B}$ points down through Path (1), in a negative direction, the additional flux is negative and we have

$$\Delta\Phi_B = -B(\Delta A_\perp) = -B(h\Delta x) \qquad (28)$$

Dividing Equation (28) through by Δt, and taking the limit that Δt goes to dt, gives

$$\frac{\Delta\Phi_B}{\Delta t} = -Bh\frac{\Delta x}{\Delta t}$$

$$\frac{d\Phi_B}{dt} = -Bh\frac{dx}{dt} = -Bhv \qquad (29)$$

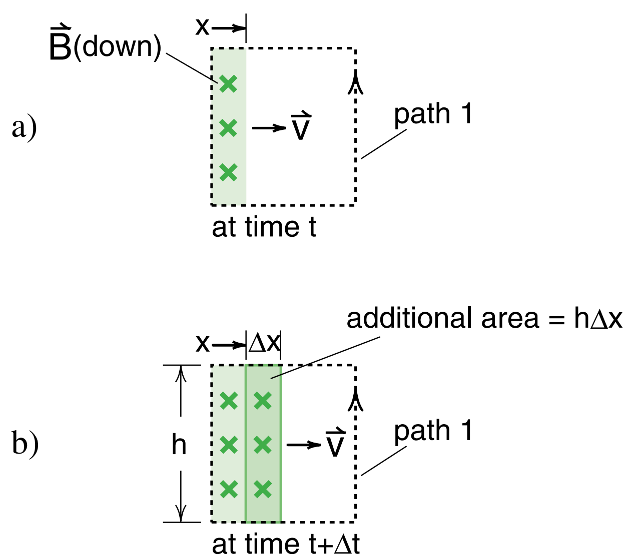


Figure 18
As the front expands, there is more magnetic flux down through Path (1).

We now have a formula for $\vec{E}\cdot d\vec{\ell}$ (Equation 27) and for $d\Phi_E/dt$ (Equation 29) which we can substitute into Faraday's law (23b) to get

$$\oint \vec{E}\cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$$

$$Eh = -(-Bvh) = +Bhv$$

The factor of h cancels and we are left with

$$\boxed{E = Bv} \quad \textit{from Faraday's law} \qquad (30)$$

which is a surprisingly simple relationship between the strengths of the electric and magnetic fields.

Analysis of Path 2

Path (2), shown in Figure (17c), is chosen to have one side in and parallel to the magnetic field. We have gone around clockwise so that $\vec{B}$ and $d\vec{\ell}$ point in the same direction. Integrating $\vec{B}$ around the path gives

$$\oint \vec{B}\cdot d\vec{\ell} = Bh \tag{31}$$

Combining Equation 31 with Ampere's law

$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$$

gives

$$Bh = \mu_0\varepsilon_0 \frac{d\Phi_E}{dt} \tag{32}$$

To evaluate $d\Phi_E/dt$, we first note that for a clockwise path, the positive direction is down into the paper in Figure (17c). This is the same direction as the electric field, thus we have a positive electric flux through path (2).

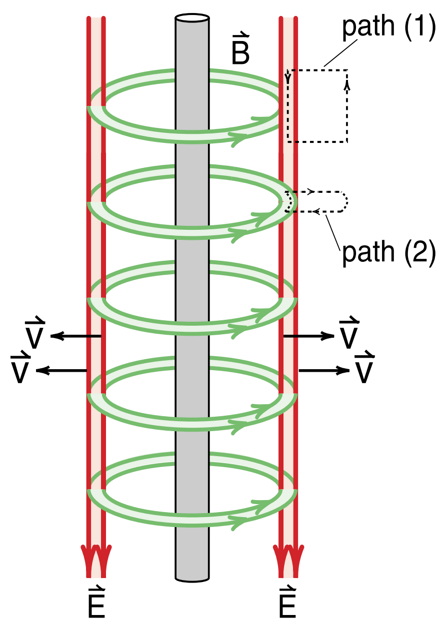


Figure 17a (repeated)
We will now turn our attention to path 2 which has one side parallel to the magnetic field.

In Figure (19), we show the expanding front at time t (19a) and at time t + Δt (19b). The increase in electric flux $\Delta\Phi_E$ is (E) times the increased area $(h\Delta x)$

$$\Delta\Phi_E = E(h\Delta x)$$

Dividing through by Δt, and taking the limit that Δt goes to dt, gives

$$\frac{\Delta\Phi_E}{\Delta t} = Eh\frac{\Delta x}{\Delta t}$$

$$\frac{d\Phi_E}{dt} = Eh\frac{dx}{dt} = Ehv \tag{33}$$

Using Equation 33 in 32, and then cancelling h, gives

$$Bh = \mu_0\varepsilon_0 Ehv$$

$$\boxed{B = \mu_0\varepsilon_0 Ev} \quad \textit{From Ampere's law} \tag{34}$$

which is another simple relationship between E and B.

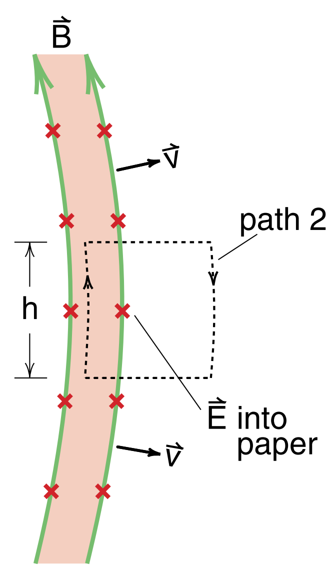


Figure 17c
An increasing (negative) electric flux flows down through Path (2).

If we divide Equation (30) $Bv = E$, by Equation (34) $B = \mu_0\varepsilon_0 Ev$, both E and B cancel giving

$$\frac{Bv}{B} = \frac{E}{\mu_0\varepsilon_0 Ev}$$

$$v^2 = \frac{1}{\mu_0\varepsilon_0}$$

$$\boxed{v = \frac{1}{\sqrt{\mu_0\varepsilon_0}}} \quad \textit{speed of light!!!} \qquad (35)$$

Thus the electromagnetic pulse of Figures (16) and (17) expands outward at the speed $1/\sqrt{\mu_0\varepsilon_0}$ which we have seen is 3×10^8 meters per second. ***Maxwell recognized that this was the speed of light and recognized that the electromagnetic pulse must be closely related to light itself.***

Using $v = 1/\sqrt{\mu_0\varepsilon_0} = c$ in Equation (34) we get

$$\boxed{B = \frac{E}{c}} \qquad (36)$$

as the relative strength of the electric and magnetic fields in an electromagnetic pulse, or as we shall see, any light wave. If we had used a reasonable set of units where c = 1 (like feet and nanoseconds), then E and B would have equal strengths in a light wave.

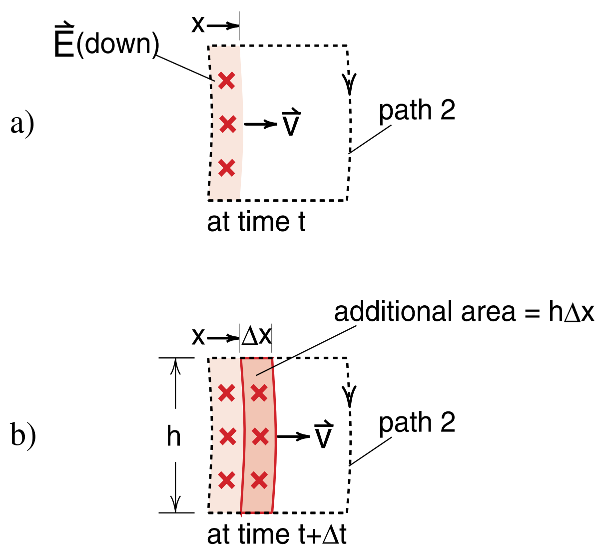


Figure 19
As the front expands, there is more electric flux down through Path (2).

Exercise 4
Construct paths like (1) and (2) of Figure (17), but which include the back side, rather than the front side, of the electromagnetic pulse. Repeat the kind of steps used to derive Equation (35) to show that the back of the pulse also travels outward at a speed $v = 1/\sqrt{\mu_0\varepsilon_0}$. As a result the pulse maintains its thickness as it expands out through space.

Exercise 5
After a class in which we discussed the electromagnetic pulse shown in Figure (20a), a student said she thought that the electric field would get ahead of the magnetic field as shown in Figure (20b). Use Maxwell's equations to show that this does not happen.

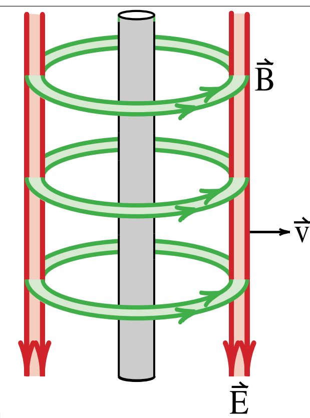


Figure 20a
The radiated electromagnetic pulse we saw in Figures (16) and (17).

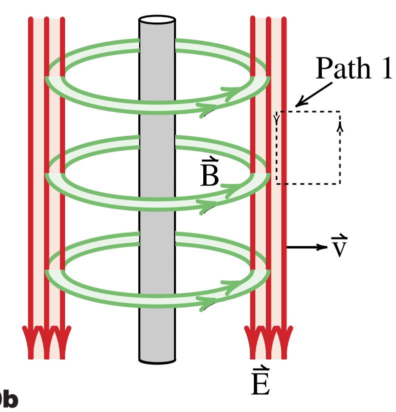


Figure 20b
The student guessed that the electric fields would get out ahead of the magnetic field. Use Path (1) to show that this does not happen.

ELECTROMAGNETIC WAVES

The single electromagnetic pulse shown in Figure (17) is an example of an electromagnetic wave. We usually think of a wave as some kind of oscillating sinusoidal thing, but as we saw in our discussion of waves on a Slinky in Chapter 1, the simplest form of a wave is a single pulse like that shown in Figure (21). The basic feature of the Slinky wave pulse was that it maintained its shape while it moved down the Slinky at the wave speed v . Now we see that the electromagnetic pulse maintains its structure of $\vec{E}$ and $\vec{B}$ fields while it moves at a speed $v = c$ through space.

We made a more or less sinusoidal wave on the Slinky by shaking one end up and down to produce a series of alternate up and down pulses that traveled together down the Slinky. Similarly, if we use an alternating current in the wire of Figure (17), we will get a series of electromagnetic pulses that travel out from the wire. This series of pulses will more closely resemble what we usually think of as an electromagnetic wave.

Figure (22a) is a graph of a rather jerky alternating current where we turn on an upward directed current of magnitude i_0, then shut off the current for a while, then turn on a downward directed current i_0, etc. This series of current pulse produces the series of electromagnetic pulses shown in Figure (22b). Far out from the wire where we can neglect the curvature of the magnetic field, we see a series of pulses shown in the close-up view, Figure (23a). This series of flat or non-curved pulses is called a ***plane wave*** of electromagnetic radiation.

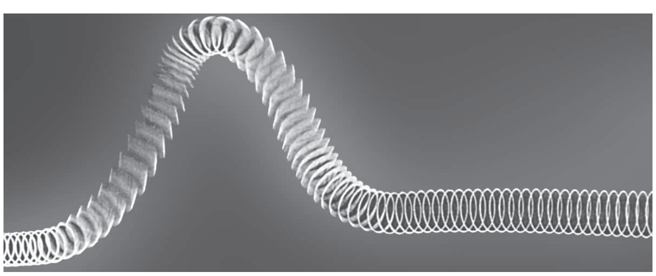


Figure 21
Slinky wave pulse.

If we used a sinusoidally oscillating current in the wire of Figure (22), then the series of electromagnetic pulses would blend together to form the sinusoidally varying electric and magnetic fields structure shown in Figure (23b). This is the wave structure one usually associates with an electromagnetic wave.

When you think of an electromagnetic wave, picture the fields shown in Figure (23), moving more or less as a rigid object past you at a speed c. The distance λ between crests is called the ***wavelength*** of the wave. The time T it takes one wavelength or cycle to pass you is

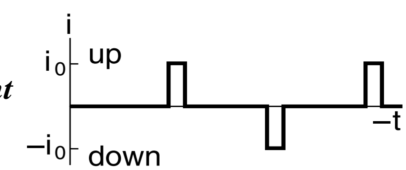
$$T \frac{\text{second}}{\text{cycle}} = \frac{\lambda \frac{\text{meter}}{\text{cycle}}}{c \frac{\text{meter}}{\text{second}}} = \frac{\lambda}{c} \frac{\text{second}}{\text{cycle}} \tag{37}$$

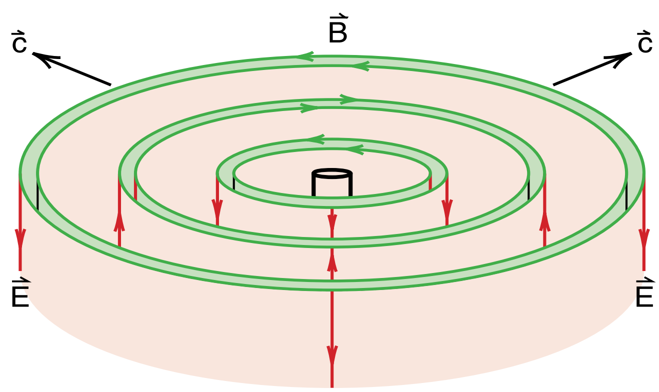
T is called the ***period*** of the wave. The ***frequency*** of the wave, the number of wavelengths or full cycles of the wave that pass you per second is

$$f \frac{\text{cycle}}{\text{second}} = \frac{c \frac{\text{meter}}{\text{second}}}{\lambda \frac{\text{meter}}{\text{cycle}}} = \frac{\lambda}{c} \frac{\text{cycle}}{\text{second}} \tag{38}$$

In Equations (37) and (38) we gave λ the dimensions meters/cycle, T of seconds/cycle and f of cycles/second so that we can use the dimensions to remember the

a) Graph of current pulses in wire





b) Resulting electric and magnetic fields

Figure 22
Fields produced by a series of current pulses.

formulas $T = \lambda/c$, $f = c/\lambda$. (It is now common to use "hertz" or "Hz" for the dimensions of frequency. This is a classic example of ruining simple dimensional analysis by using people's names.) Finally, the angular frequency ω radians per second is defined as

$$\omega\frac{\text{radians}}{\text{sec ond}} = 2\pi\frac{\text{radians}}{\text{cycle}}\times f\frac{\text{cycles}}{\text{second}}$$
$$= 2\pi f\frac{\text{radians}}{\text{second}} \qquad (39)$$

You can remember where the 2π goes by giving it the dimensions 2π radians/cycle. (Think of a full circle or full cycle as having 2π radians.) We will indiscriminately use the word *frequency* to describe either f cycles/second or ω radians/second, whichever is more appropriate. If, however, we say that something has a frequency of so many hertz, as in 60 Hz, we will always mean cycles/second.

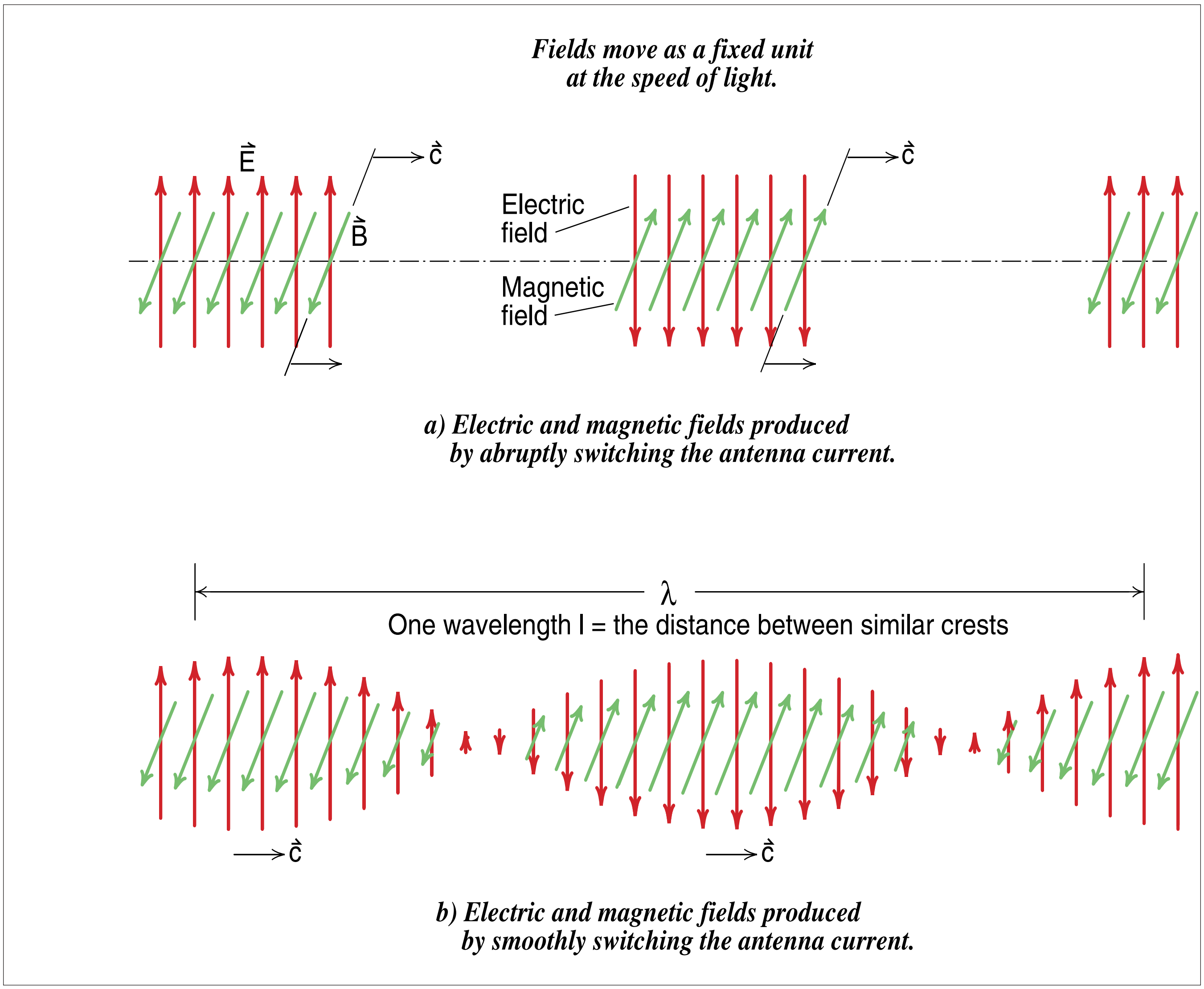


Figure 23
Structure of electric and magnetic fields in light and radio waves.

ELECTROMAGNETIC SPECTRUM

We have seen by direct calculation that the electromagnetic pulse of Figure (17), and the series of pulses in Figure (22) are a solution of Maxwell's equations. It is not much of an extension of our work to show that the sinusoidal wave structure of Figure (23b) is also a solution. The fact that all of these structures move at a speed $c = 1/\sqrt{\mu_0\varepsilon_0} = 3\times10^8$ m/s is what suggested to Maxwell that these electromagnetic waves were light, that he had discovered the theory of light.

But there is nothing in Maxwell's equations that restricts our sinusoidal solution in Figure (23b) to certain values or ranges of frequency or wavelength. One hundred years before Maxwell it was known from interference experiments (which we will discuss in the next chapter) that light had a wave nature and that the wavelengths of light ranged from about 6×10^{-5}cm in the red part of the spectrum down to 4×10^{-5}cm in the blue part. With the discovery of Maxwell's theory of light, it became clear that there must be a complete spectrum of electromagnetic radiation, from very long down to very short wavelengths, and that visible light was just a tiny piece of this spectrum.

More importantly, Maxwell's theory provided the clue as to how you might be able to create electromagnetic waves at other frequencies. We have seen that an oscillating current in a wire produces an electromagnetic wave whose frequency is the same as that of the current. If, for example, the frequency of the current is 1030 kc (1030 kilocycles) = 1.03×10^6 cycles/sec, then the electromagnetic wave produced should have a wavelength

$$\lambda\,\frac{\text{meters}}{\text{cycle}} = \frac{c\,\dfrac{\text{meters}}{\text{second}}}{f\,\dfrac{\text{cycles}}{\text{sec ond}}} = \frac{3\times10^8\text{m/s}}{103\times10^6\text{c/s}}$$

$$= 297 \text{ meters}$$

Such waves were discovered within 10 years of Maxwell's theory, and were called radio waves. The frequency 1030 kc is the frequency of radio station WBZ in Boston, Mass.

Components of the Electromagnetic Spectrum

Figure (24) shows the complete electromagnetic spectrum as we know it today. We have labeled various components that may be familiar to the reader. These components, and the corresponding range of wavelengths are as follows:

Radio Waves	10^6 m to .05 mm
AM Band	500 m to 190 m
Short Wave	60 m to 15 m
TV VHF Band	10 m to 1 m
TV UHF Band	1 m to 10 cm
Microwaves	10 cm to .05 mm
Infrared Light	.05 mm to 6×10^{-5} cm
Visible Light	6×10^{-5} to 4×10^{-5}cm
Ultraviolet Light	4×10^{-5} cm to 10^{-6}cm
X Rays	10^{-6} cm to 10^{-9} cm
γ Rays	10^{-9} cm and shorter

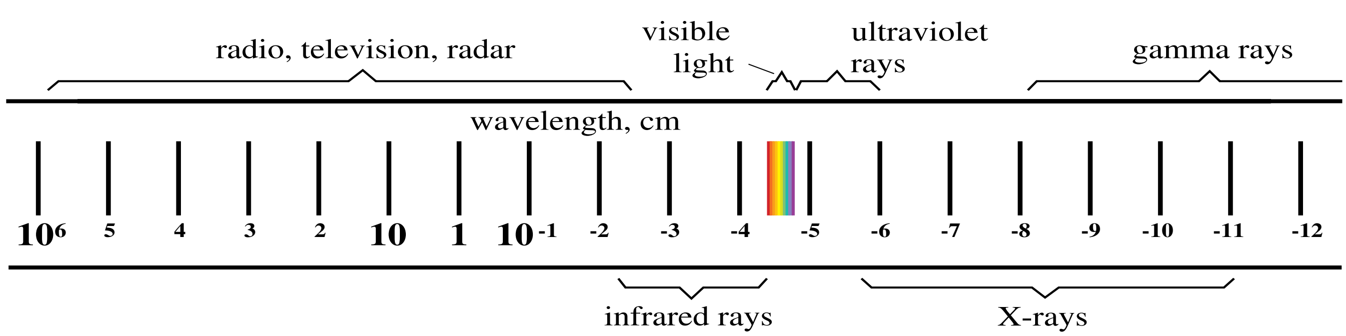


Figure 24
The electromagnetic spectrum extends from long wavelength radio waves down to short wavelength X rays and gamma rays. The visible part of the spectrum is indicated by the small box.

In each of these ranges, the most efficient way to emit or detect the radiation is to use antennas whose size is comparable to the wavelength of the radiation. For radio waves the antennas are generally some kind of a structure made from wire. In the infrared and the visible region, radiation is generally emitted by molecules and atoms. The short wavelength x rays and γ rays generally come from atomic nuclei or subatomic particles.

The longest wavelength radio waves that have been studied are the so-called "whistlers", radio waves with an audio frequency, that are produced by lightening bolts and reflected back and forth around the earth by charged particles trapped in the earth's magnetic field. On a shorter scale of distance are the long wavelength radio waves which penetrate the ocean and are used for communications with submarines. The radio station in Cutler, Maine, shown in Figure (25), has twenty-six towers over 1000 feet tall to support the antenna to produce such waves. This station, operated by the United States Navy, is the world's most powerful.

As we go to shorter wavelengths and smaller antennas, we get to the broadcast band, short wave radio, then to the VHF and UHF television frequencies. (FM radio is tucked into the VHF band next to Channel 6). The wavelengths for VHF television are of the order of meters, while those for UHF are of the order of a foot. Those with separate VHF and UHF television antennas will be familiar with the fact that the UHF antenna, which detects the shorter wavelengths, is smaller in size.

Adjusting the rabbit ears antenna on a television set provides practical experience with the problems of detecting an electromagnetic wave. As the TV signal strikes the antenna, the electric field in the wave acts on the electrons in the TV antenna wire. If the wire is parallel to the electric field, the electrons are pushed along in the wire producing a voltage that is detected by the television set. If the wire is perpendicular, the electrons will not be pushed up and down and no voltage will be produced.

The length of the wire is also important. If the antenna were one half wavelength, then the electric field at one end would be pushing in the opposite direction from the field at the other end, the integral $\int \vec{E} \cdot \vec{dl}$ down the antenna would be zero, and you would get no net voltage or signal. You want the antenna long enough to get a big voltage, but not so long that the electric field in one part of the antenna works against the field in another part. One quarter wavelength is generally the optimum antenna length.

Figure 25
The worlds largest radio station at Cutler, Maine. This structure, with 75 miles of antenna wire and 26 towers over 1000 ft high, generates long wavelength low frequency, radio waves for communications with submarines.

The microwave region, now familiar from microwave communications and particularly microwave ovens, lies between the television frequencies and infrared radiation. The fact that you heat food in a microwave oven emphasizes the fact that electromagnetic radiation carries energy. One can derive that the energy density in an electromagnetic wave is given by the formula

$$\left.\begin{array}{l}\text{energy density in an}\\ \text{electromagnetic}\\ \text{wave}\end{array}\right\} = \frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \qquad (37)$$

We have already seen the first term $\varepsilon_0 E^2/2$, when we calculated the energy stored in a capacitor (see Equation 27-36 on page 27-19). If we had calculated the energy to start a current in an inductor, we would have gotten the formula $B^2/2\mu_0$ for the energy density in that device. Equation (37) tells us the amount of energy is associated with electric and magnetic fields whenever we find them.

Blackbody Radiation

Atoms and molecules emit radiation in the infrared, visible and ultraviolet part of the spectrum. One of the main sources of radiation in this part of the spectrum is the so-called ***blackbody*** radiation emitted by objects due to the thermal motion of their atoms and molecules.

If you heat an iron poker in a fire, the poker first gets warm, then begins to glow a dull red, then a bright red or even, orange. At higher temperatures the poker becomes white, like the filaments in an electric light bulb. At still higher temperatures, if the poker did not melt, it would become bluish. The name *blackbody radiation* is related to the fact that an initially cold, black object emits these colors of light when heated.

There is a well studied relationship between the temperature of an object and the predominant frequency of the blackbody radiation it emits. Basically, the higher the temperature, the higher the frequency. Astronomers use this relationship to determine the temperature of stars from their color. The infrared stars are quite cool, our yellow sun has about the same temperature as the yellow filament in an incandescent lamp, and the blue stars are the hottest.

All objects emit blackbody radiation. You, yourself, are like a small star emitting infrared radiation at a wavelength corresponding to a temperature of 300K. In an infrared photograph taken at night, you would show up distinctly due to this radiation. Infrared photographs are now taken of houses at night to show up hot spots and heat leaks in the house.

Perhaps the most famous example of blackbody radiation is the 3K cosmic background radiation which is the remnant of the big bang which created the universe. We will say much more about this radiation in Chapter 34.

UV, X Rays, and Gamma Rays

When we get to wavelengths shorter than the visible spectrum, and even in the visible spectrum, we begin to run into problems with Maxwell's theory of light. These problems were first clearly displayed by Max Planck who in 1900 developed a theory that explained the blackbody spectrum of radiation. The problem with Planck's theory of blackbody radiation is that it could not be derived from Maxwell's theory of light and Newtonian mechanics. His theory involved arbitrary assumptions that would not be understood for another 23 years, until after the development of quantum mechanics.

Despite the failure of Newton's and Maxwell's theories to explain all the details, the electromagnetic spectrum continues right on up into the shorter wavelengths of ultraviolet (UV) light, then to x rays and finally to γ rays. Ultraviolet light is most familiar from the effect it has on us, causing tanning, sunburns, and skin cancer depending on the intensity and duration of the dose. The ozone layer in the upper atmosphere, as long as it lasts, is important because it filters out much of the ultraviolet light emitted by the sun.

X rays are famous for their ability to penetrate flesh and produce photographs of bones. These rays are usually emitted by the tightly bound electrons on the inside of large atoms, and also by nuclear reactions. The highest frequency radiation, γ rays, are emitted by the smallest objects—nuclei and elementary particles.

POLARIZATION

One of the immediate tests of our picture of a light or radio wave, shown in Figure (23), is the phenomena of ***polarization***. We mentioned that the reason that you had to adjust the angle of the wires on a rabbit ears antenna was that the electric field of the television signal had to have a significant component parallel to the wires in order to push the electrons up and down the wire. Or, in the terminology of the last few chapters, we needed the parallel component of $\vec{E}$ so that the voltage $V = \oint \vec{E} \cdot \vec{dl}$ would be large enough to be detected by the television circuitry. (In this case, the line integral $\oint \vec{E} \cdot \vec{dl}$ is along the antenna wire.) Polarization is a phenomena that results from the fact that the electric field $\vec{E}$ in an electromagnetic wave can have various orientations as the wave moves through space.

Although we have derived the structure of an electromagnetic wave for the specific case of a wave produced by an alternating current in a long, straight wire, some of the general features of electromagnetic waves are clearly present in our solution. The general features that are present in all electromagnetic waves are:

1) All electromagnetic waves are a structure consisting of an electric field $\vec{E}$ and a magnetic field $\vec{B}$.

2) $\vec{E}$ and $\vec{B}$ are at right angles to each other as shown in Figure (23).

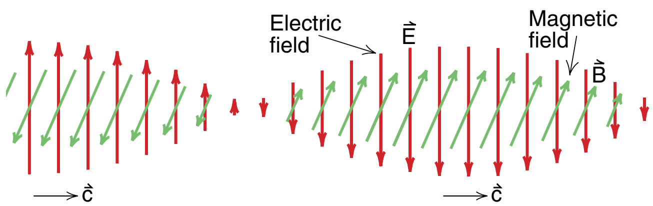
3) The wave travels in a direction perpendicular to the plane of $\vec{E}$ and $\vec{B}$.

4) The speed of the wave is $c = 3\times10^8$ m/s.

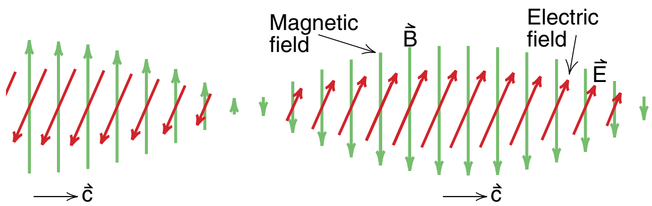
5) The relative strengths of $\vec{E}$ and $\vec{B}$ are given by Equation (36) as $B = E/c$.

Even with these restrictions, and even if we consider only flat or plane electromagnetic waves, there are still various possible orientations of the electric field as shown in Figure (26). In Figure (26a) we see a plane wave with a vertical electrical field. This would be called a vertically polarized wave. In Figure (26b), where the electric field is horizontal, we have a horizontally polarized wave. By convention we say that the direction of polarization is the direction of the electric field in an electromagnetic wave.

Because $\vec{E}$ must lie in the plane perpendicular to the direction of motion of an electromagnetic wave, **$\vec{E}$ has only two independent components**, which we can call the vertical and horizontal polarizations, or the x and y polarizations as shown in Figures (27a) and (27b) respectively. If we happen to encounter an electromagnetic wave where $\vec{E}$ is neither vertical or horizontal, but at some angle θ, we can decompose $\vec{E}$ into its x and y components as shown in (27c). Thus we can consider a wave polarized at an arbitrary angle θ as a mixture of the two independent polarizations.



a) Vertically polarized electromagnetic wave.



b) Horizontally polarized electromagnetic wave.

Figure 26
Two possible polarizations of an electromagnetic wave.

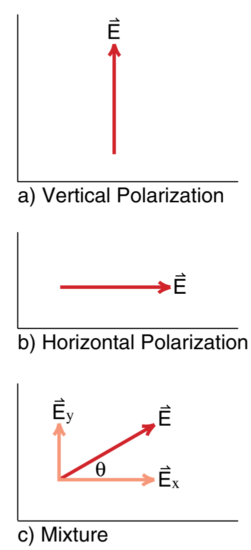


Figure 27
We define the direction of polarization of an electromagnetic wave as the direction of the electric field.

Polarizers

A polarizer is a device that lets only one of the two possible polarizations of an electromagnetic wave pass through. If we are working with microwaves whose wavelength is of the order of a few centimeters, a frame strung with parallel copper wires, as seen in Figure (28), makes an excellent polarizer. If a vertically polarized wave strikes this vertical array of wires, the electric field $\vec{E}$ in the wave will be parallel to the wires. This parallel $\vec{E}$ field will cause electrons to move up and down in the wires, taking energy out of the incident wave. As a result the vertically polarized wave cannot get through. (One can observe that the wave is actually reflected by the parallel wires.)

If you then rotate the wires 90º, so that the $\vec{E}$ field in the wave is perpendicular to the wires, the electric field can no longer move electrons along the wires and the wires have no effect. The wave passes through without attenuation.

If you do not happen to know the direction of polarization of the microwave, put the polarizer in the beam and rotate it. For one orientation the microwave beam will be completely blocked. Rotate the polarizer by 90° and you will get a maximum transmission.

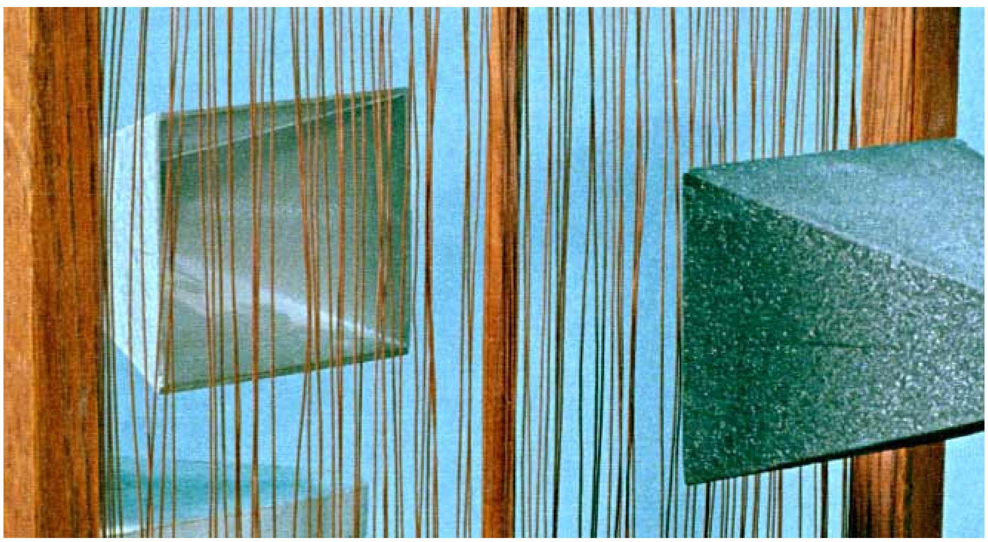


Figure 28
Microwave polarizer, made from an array of copper wires. The microwave transmitter is seen on the other side of the wires, the detector is on this side. When the wires are parallel to the transmitted electric field, no signal is detected. Rotate the wires 90 degrees, and the full signal is detected.

Light Polarizers

We can picture light from the sun as a mixture of light waves with randomly oriented polarizations. (The $\vec{E}$ fields are, of course, always in the plane perpendicular to the direction of motion of the light wave. Only the angle in that plane is random.) A polarizer made of an array of copper wires like that shown in Figure (28), will not work for light because the wavelength of light is so short $(\lambda \approx 5 \times 10^{-5}\,\text{cm})$ that the light passes right between the wires. For such a polarizer to be effective, the spacing between the wires would have to be of the order of a wavelength of light or less.

A polarizer for light can be constructed by imbedding long-chain molecules in a flexible plastic sheet, and then stretching the sheet so that the molecules are aligned parallel to each other. The molecules act like the wires in our copper wire array, but have a spacing of the order of the wavelength of light. As a result the molecules block light waves whose electric field is parallel to them, while allowing waves with a perpendicular electric field to pass. (The commercial name for such a sheet of plastic is *Polaroid*.)

Since light from the sun or from standard electric light bulbs consists of many randomly polarized waves, a single sheet of Polaroid removes half of the waves no matter how we orient the Polaroid (as long as the sheet of Polaroid is perpendicular to the direction of motion of the light beam). But once the light has gone through one sheet of Polaroid, all the surviving light waves have the same polarization. If we place a second sheet of Polaroid over the first, all the light will be absorbed if the long molecules in the second sheet are perpendicular to the long molecules in the first sheet. If the long molecules in the second sheet are parallel to those in the first, most of the waves that make it through the first, make it through the second also. This effect is seen clearly in Figure (29).

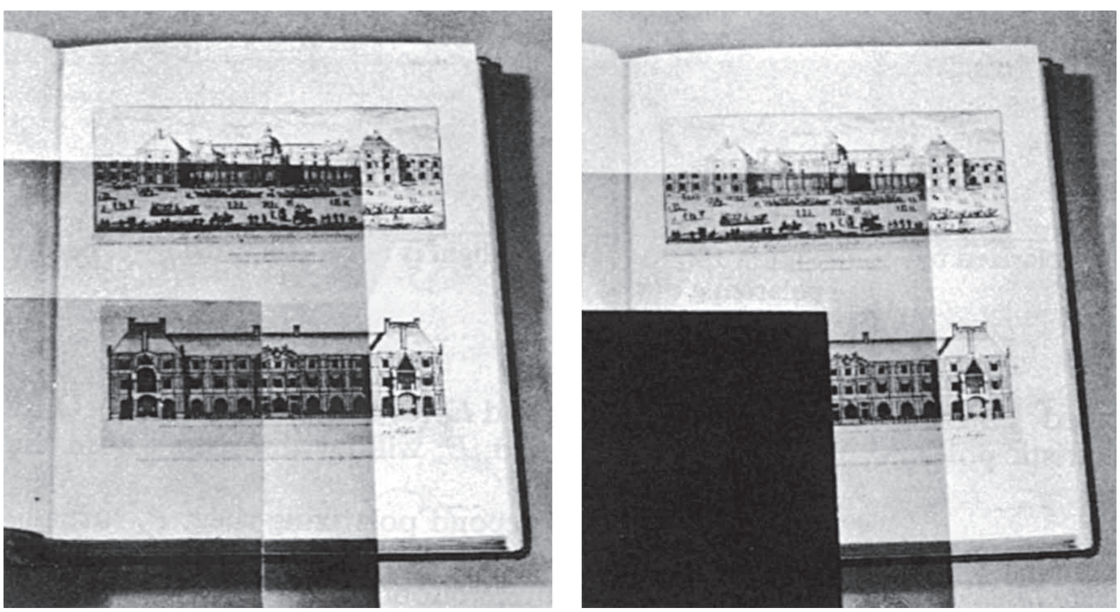


Figure 29
Light polarizers. Two sheets of polaroid are placed on top of a drawing. On the left, the axes of the sheets are parallel, so that nearly half the light passes through. On the right, the axes are perpendicular, so that no light passes through. (Photo from Halliday & Resnick)

Magnetic Field Detector

So far, our discussion of electromagnetic radiation has focused primarily on detecting the ***electric field*** in the wave. The rabbit ear antenna wire had to be partially parallel to the electric field so that $\oint \vec{E}\cdot d\vec{\ell}$ and therefore the voltage on the antenna would not be zero. In our discussion of polarization, we aligned the parallel array of wires or molecules parallel to the electric field when we wanted the radiation to be reflected or absorbed.

It is also fairly easy to detect the **magnetic field** in a radio wave by using one of our $\oint \vec{E}\cdot d\vec{\ell}$ meters to detect a changing magnetic flux (an application of Faraday's law). This is the principle behind the radio direction finders featured in a few World War II spy pictures.

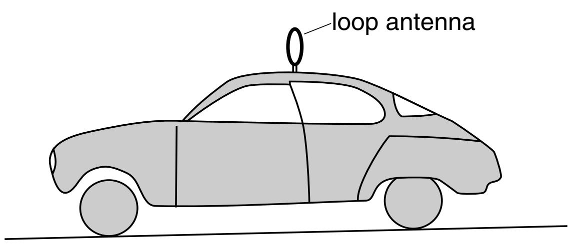


Figure 30a
Car with radio direction finder loop antenna mounted on top.

In a typical scene we see a car with a metal loop mounted on top as shown in Figure (30a). It is chasing another car with a hidden transmitter, or looking for a clandestine enemy transmitter.

If the transmitter is a radio antenna with a vertical transmitting wire as shown in Figure (30b), the magnetic field of the radiated wave will be concentric circles as shown. Objects on the ground, the ground itself, and nearby buildings and hills can distort this picture, but for now we will neglect the distortions.

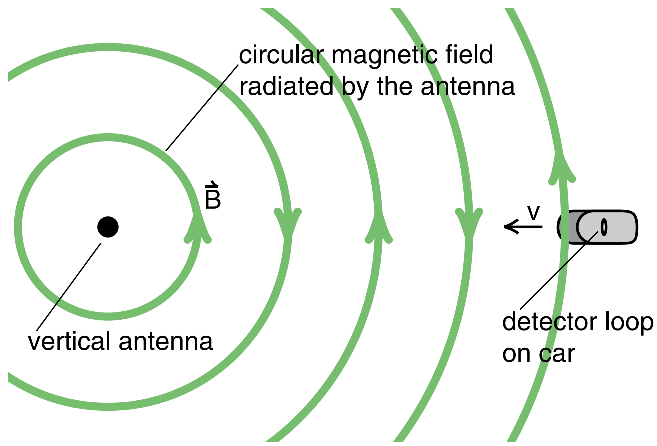


Figure 30b
Car driving toward radio transmitter.

Figure 31
In a January 1998 National Geographic article on Amelia Earhardt, there appeared a picture of a vintage Electra airplane similar to the one flown by Earhardt on her last trip in 1938. On the top of the plane, you can see the kind of radio direction finder we have been discussing. (The plane is being flown by Linda Finch.)

In Figures (32a) and (32b), we show the magnetic field of the radio wave as it passes the detector loop mounted on the car. A voltmeter is attached to the loop as shown in Figure (33). In (32a), the plane of the loop is parallel to $\vec{B}$, the magnetic flux Φ_B through the loop is zero, and Faraday's law gives

$$V = \oint \vec{E}\cdot d\vec{\ell} = \frac{d\Phi_B}{dt} = 0$$

In this orientation there is no voltage reading on the voltmeter attached to the loop.

In the orientation of Figure (32b), the magnetic field passes through the loop and we get a maximum amount of magnetic flux Φ_B. As the radio wave passes by the loop, this flux alternates signs at the frequency of the wave, therefore the rate of change of flux $d\Phi_B/dt$ is at a maximum. In this orientation we get a maximum voltmeter reading.

The most sensitive way to use this radio direction finder is to get a zero or "null" reading on the voltmeter. Only when the loop is oriented as in Figure (32a), with its plane perpendicular to the direction of motion of the radio wave, will we get a null reading. At any other orientation some magnetic flux will pass through the loop and we get some voltage.

Spy pictures, set in more modern times, do not show antenna loops like that in Figure (30) because modern radio direction finders use so-called "ferrite" antennas that detect the electric field in the radio wave. We get a voltage on a ferrite antenna when the electric field in the radio wave has a component along the ferrite rod, just as it needed a component along the wires of a rabbit ears antenna. Again these direction finders are most accurate when detecting a null or zero voltage. This occurs only when the rod is parallel to the direction of motion of the radio wave, i.e. points toward the station. (This effect is very obvious in a small portable radio. You will notice that the reception disappears and you get a null detection, for some orientations of the radio.)

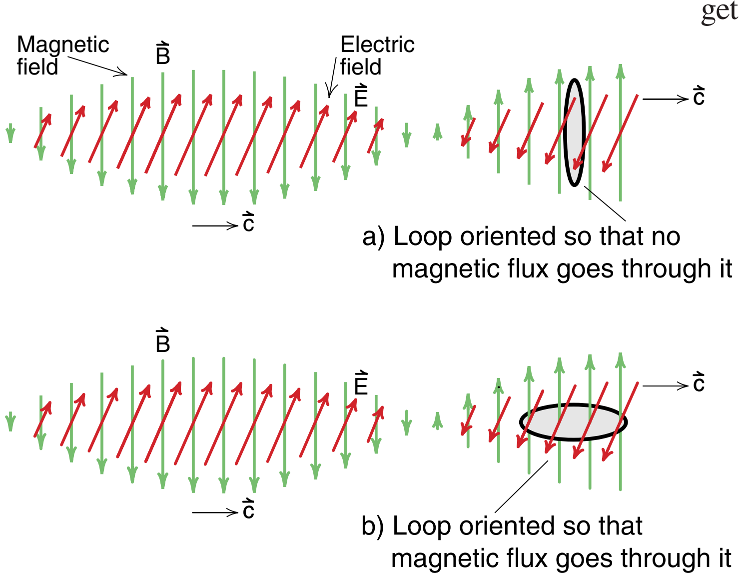


Figure 32
Electromagnetic field impinging upon a loop antenna. In (a), the magnetic field is parallel to the plane of the loop, and therefore no magnetic flux goes through the loop. In (b), the magnetic flux goes through the loop. As the wave passes by, the amount of flux changes, inducing a voltage in the loop antenna.

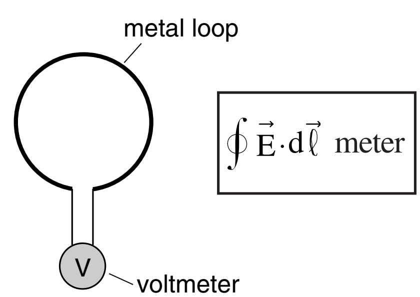


Figure 33
We can think of a wire loop connected to a voltmeter as an $\oint \vec{E}\cdot d\vec{\ell}$ meter. Any changing magnetic flux through the loop induces a voltage around the loop. This voltage is read by the voltmeter.

RADIATED ELECTRIC FIELDS

One of the best computer simulations of physical phenomena is the series of short films about the electric fields produced by moving and accelerated charges. We will describe a few of the frames from these films, but nothing replaces watching them.

Two basic ideas underlie these films. One is Gauss' law which requires that ***electric field lines not break***, do not end, in empty space. The other is that disturbances on an ***electric field line travel outward at the speed of light***. No disturbance, no change in the electric field structure, can travel faster than the speed of light without violating causality. (You could get answers to questions that have not yet been asked.)

As an introduction to the computer simulations of radiation, let us see how a simple application of these two basic ideas leads to the picture of the electromagnetic pulse shown back in Figure (16). In Figure (34a) we show the electric field of a stationary, positively charged rod. The electric field lines go radially outward to infinity. (It's a long rod, and it has been at rest for a long time.)

At time $t = 0$ we start moving the entire rod upward at a speed v. By Gauss' law the electric field lines must stay attached to the charges Q in the rod, so that the ends of the electric field lines have to start moving up with the rod.

No information about our moving the rod can travel outward from the rod faster than the speed of light. If the time is now $t > 0$, then beyond a distance ct, the electric field lines must still be radially outward as in Figure (34b). To keep the field lines radial beyond $r = ct$, and keep them attached to the charges +Q in the rod, there must be some kind of expanding kink in the lines as indicated.

At time $t = t_1$, we stop moving the positively charged rod. The information that the charged rod has stopped moving cannot travel faster than the speed of light, thus the displaced radial field next to the rod cannot be any farther out than a distance $c(t - t_1)$ as shown in Figure (34c). The effect of starting, then stopping the positive rod is an outward traveling kink in the electric field lines. It is as if we had ropes attached to the positive rod, and jerking the rod produced an outward traveling kink or wave on the ropes.

In Figure (34d), we have added in a stationary negatively charged rod and the inward directed electric field produced by that rod. The charge density on the negative rod is opposite that of the positive rod, so that there is no net charge on the two rods. When we combine these rods, all we have left is a positive upward directed current during the time interval $t = 0$ to $t = t_1$. We have a short current pulse, and the electric field produced by the current pulse must be the vector sum of the electric fields of the two rods.

In Figure (34e), we add up the two electric fields. In the region $r > ct$ beyond the kink, the positive and negative fields must cancel exactly. In the region $r < (t - t_1)$ we should also have nearly complete cancellation. Thus all we are left with are the fields $\vec{E}_+$ and $\vec{E}_-$ inside the kink as shown in Figure (34f). Since electric field lines cannot end in empty space, $\vec{E}_+$ and $\vec{E}_-$ must add up to produce the downward directed $\vec{E}_{net}$ shown in Figure (34g). Note that this downward directed electric field pulse was produced by an upward directed current pulse. As we have seen before, this induced electric field opposes the change in current.

In Figure (34h) we added the expanding magnetic field pulse that should be associated with the current pulse. What we see is an expanding electromagnetic pulse that has the structure shown in Figure (16). Simple arguments based on Gauss' law and causality gave us most of the results we worked so hard to get earlier. What we did get earlier, however, when we applied Ampere's and Faraday's law to this field structure, was the explicit prediction that the pulse expands at the speed $1/\sqrt{\mu_0\varepsilon_0} = 3 \times 10^8$ m/s.

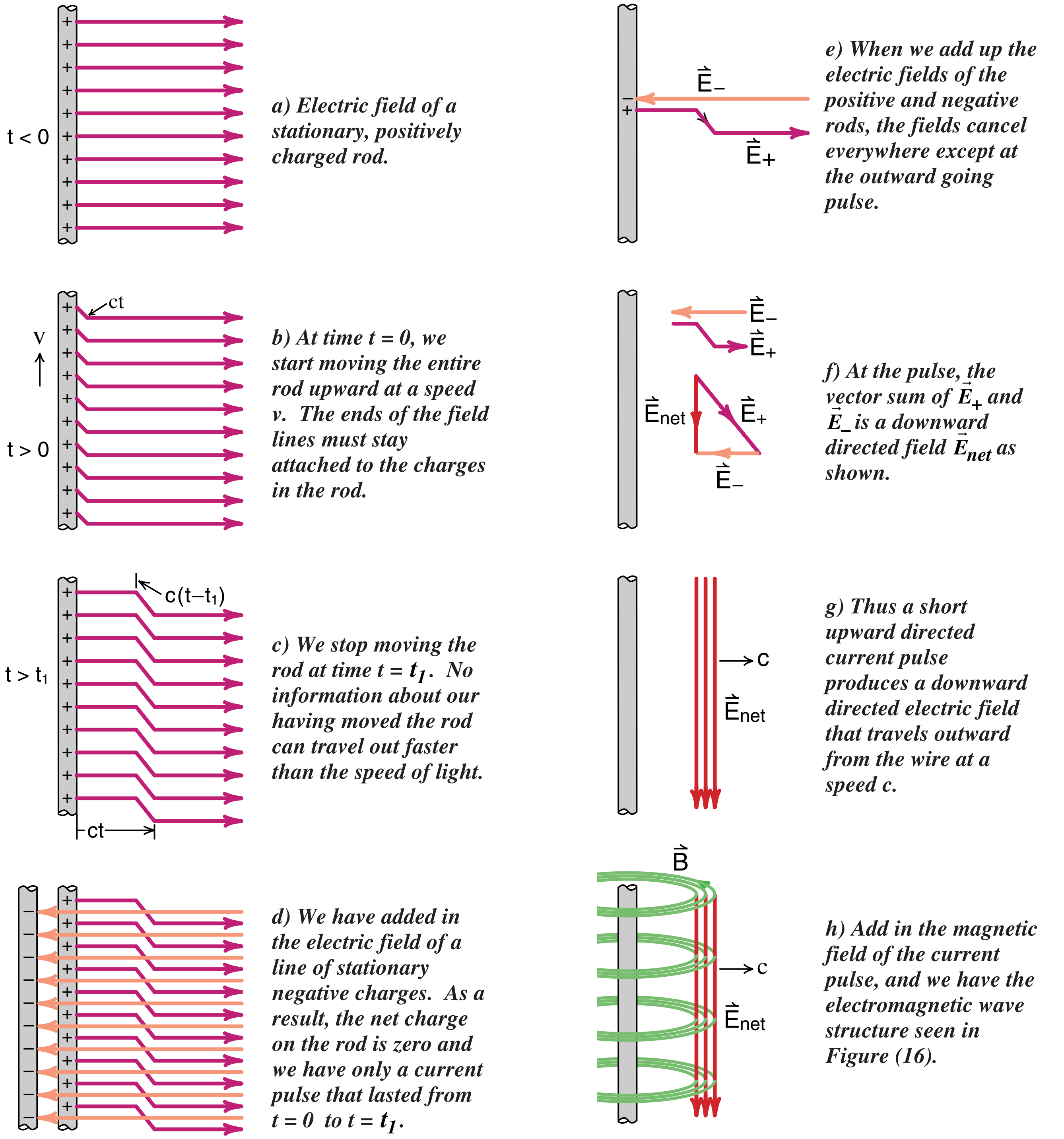


Figure 34
Using the fact that electric field lines cannot break in empty space (Gauss' Law), and the idea that kinks in the field lines travel at the speed of light, we can guess the structure of an electromagnetic pulse.

Field of a Point Charge

The computer simulations show the electric field of a point charge under varying situations. In the first, we see the electric field of a point charge at rest, as shown in Figure (35a). Then we see a charge moving at constant velocity $\vec{v}$. As the speed of the charge approaches c, the electric field scrunches up as shown in Figure (35b).

The next film segment shows what happens when we have a moving charge that stops. If the charge stopped at time $t = 0$, then at a distance $r = ct$ or greater, we must have the electric field of a moving charge, because no information that the charge has stopped can reach beyond this distance. In close we have the electric field of a static charge. The expanding kink that connects the two regions is the electromagnetic wave. The result is shown in Figure (35c). The final film segment shows the electric field of an oscillating charge. Figure (36) shows one frame of the film. This still picture does a serious injustice to the animated film. There is no substitute, or words to explain, what you see and feel when you watch this film.

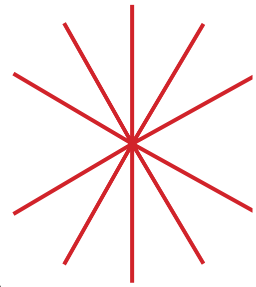


Figure 35a
Electric field of a stationary charge.

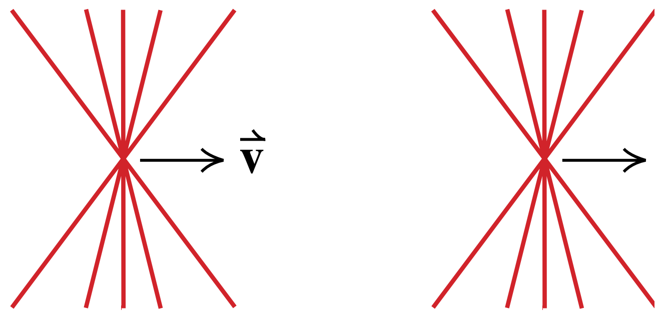


Figure 35b
Electric field of a moving charge. If the charge has been moving at constant speed for a long time, the field is radial, but squeezed up at the top and bottom.

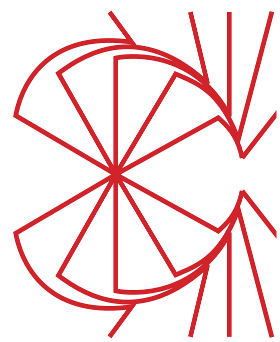


Figure 35c
Field of a charge that stopped. Assume that the charge stopped t seconds ago. Inside a circle of radius ct, we have the field of a stationary charge. Outside, where there is no information that the charge has stopped, we still have the field of a moving charge. The kink that connects the two fields is the electromagnetic radiation.

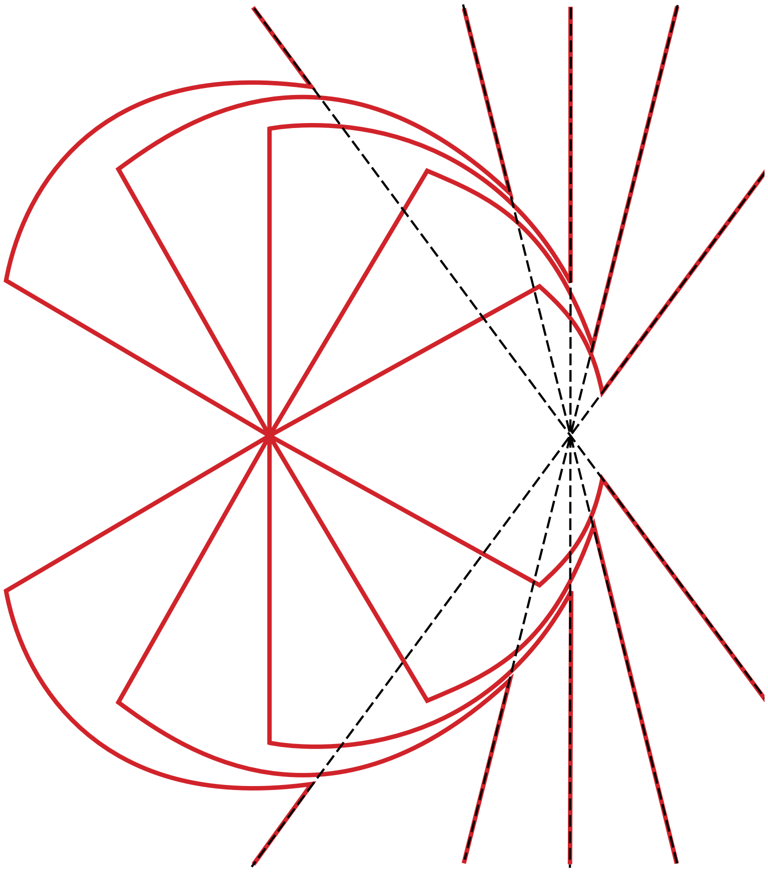


Figure 35c (enlarged)
Electric field of a charge that stopped. The dotted lines show the field structure we would have seen had the charge not stopped.

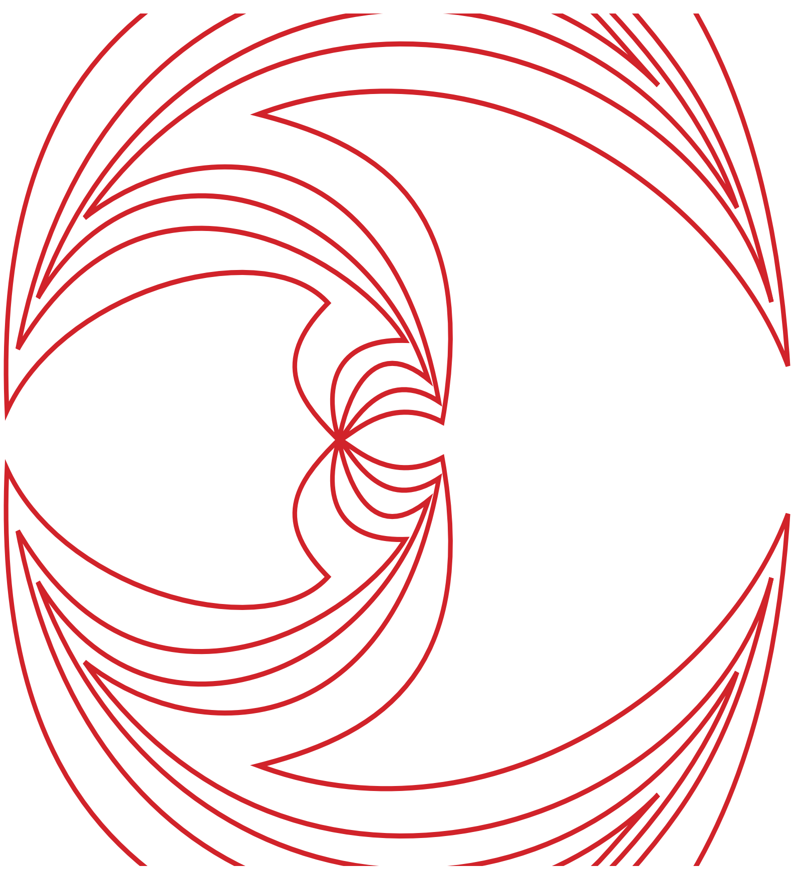


Figure 36
Electric field of an oscillating charge.

Exercise 6

Assume that we have a supply of ping pong balls and cardboard tubes shown in Figures (37). By looking at the fields outside these objects decide what could be inside producing the fields. Explicitly do the following for each case.

i) Write down the Maxwell equation which you used to decide what is inside the ball or tube, and explain how you used the equation.

ii) If more than one kind of source could produce the field shown, describe both (or all) sources and show the appropriate Maxwell equations.

iii) If the field is impossible, explain why, using a Maxwell equation to back up your explanation.

In each case, we have indicated whether the source is in a ball or tube. Magnetic fields are dashed lines, electric fields are solid lines, and the balls and tubes are surrounded by empty space.

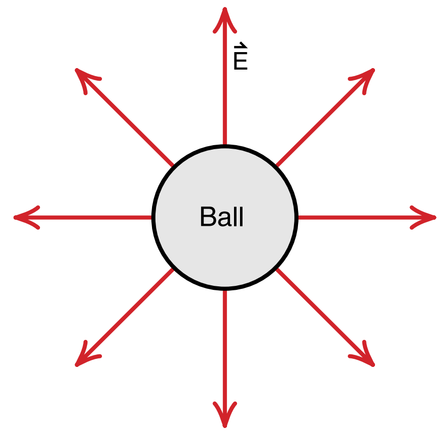


Figure 37a
Electric field emerging from ping pong ball.

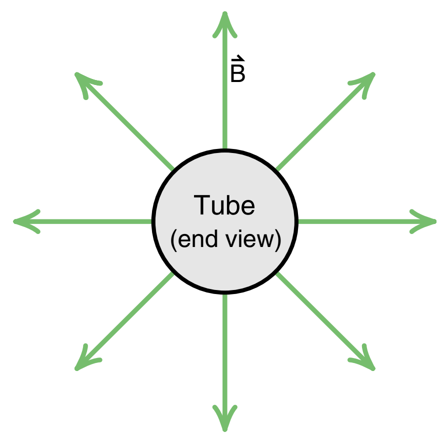


Figure 37b
Magnetic field emerging from ping pong ball.

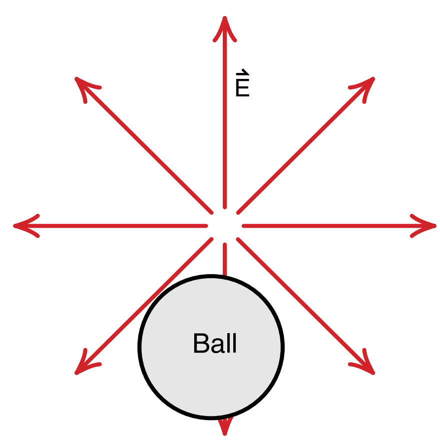


Figure 37c
Electric field emerging above ping pong ball.

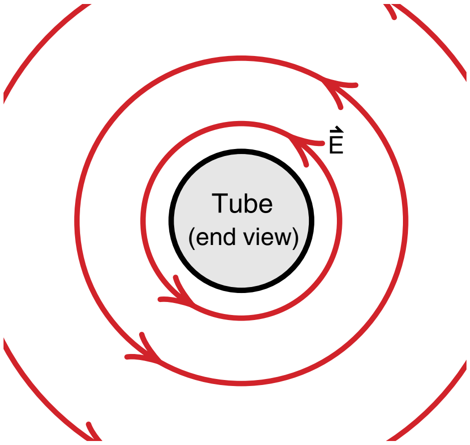


Figure 37d
Electric field around tube.

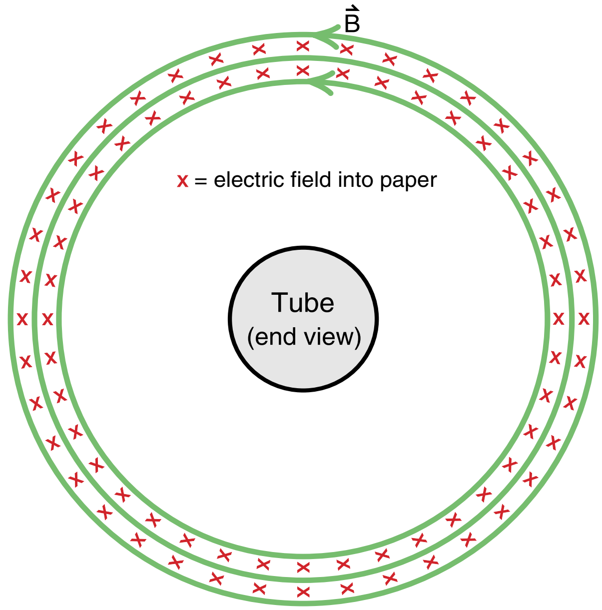


Figure 37f
For this example, explain what is happening to the fields, what is in the tube, and what happened inside.

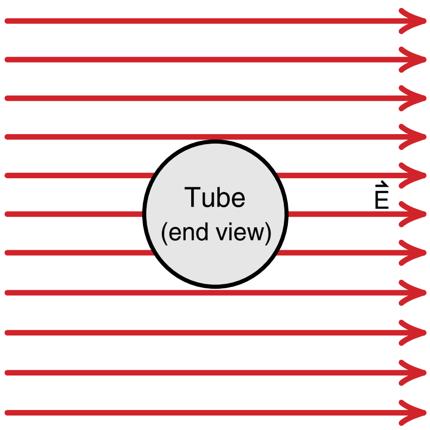


Figure 37e
Electric field passing through tube.

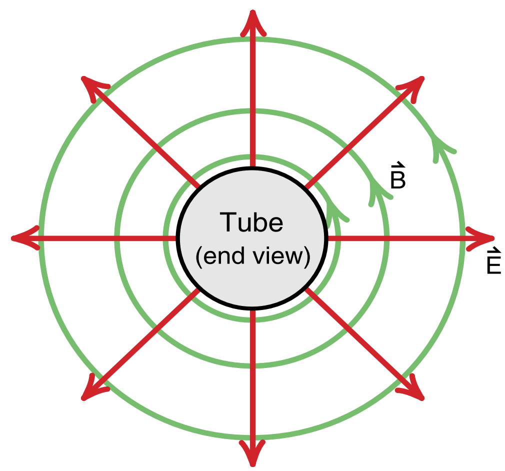


Figure 37g
There is only ONE object inside this tube. What is it? What is it doing?

Index