Oscillations and vibrations play a more significant role in our lives than we realize. When you strike a bell, the metal vibrates, creating a sound wave. All musical instruments are based on some method to force the air around the instrument to oscillate. Oscillations from the swing of a pendulum in a grandfather’s clock to the vibrations of a quartz crystal are used as timing devices. When you heat a substance, some of the energy you supply goes into oscillations of the atoms. Most forms of wave motion involve the oscillatory motion of the substance through which the wave is moving.

Despite the enormous variety of systems that oscillate, they have many features in common, features exhibited by the simple system of a mass on a spring. As a result we will focus our attention on the analysis of the motion of a mass on a spring, describing ways in which other forms of oscillation are similar.
OSCILLATORY MOTION

Suspend a mass on the end of a spring as shown in Figure (1), gently pull the mass down and let go. The mass will oscillate up and down about the equilibrium position. How do we describe the kind of motion we are looking at?

One of the best ways to see what kind of motion we are dealing with is to perform the demonstration illustrated in Figure (2a). In that demonstration, we place a rotating wheel beside an oscillating mass, and view the two objects via a TV camera set off to the side as shown. A white tape is placed around the mass, and a short stick is mounted on the rotating wheel as seen in the edge view, Figure (2b). This edge view is the one displayed by the TV camera.

The wheel is mounted on a variable speed motor, which allows us to adjust the angular velocity of the \( \omega \) of the wheel so that the wheel goes around once in precisely the same length of time it takes the mass to bob up and down once. The height of the wheel is adjusted so that when the mass is at rest at its equilibrium position, the white stripe on the mass lines up with the axis of the wheel. As a result if the stick is in a horizontal position (3 o’clock or 9 o’clock) and the mass is at rest, the stick and the white stripe line up on the television image.

Now pull the mass down so that the white stripe lines up with the lowest position of the stick the same height as the stick when the stick is at the bottom position (6 o’clock). Start the motor rotating at the correct frequency and release the mass when the stick is at the bottom. If you do this just right, some practice may be required, you will see in the television picture that the white stripe and the stick move up and down together as if they were a single object.

From this demonstration we conclude that the up and down oscillatory motion of the mass is the same as circular motion viewed sideways. As a result we can use what we know about circular motion to understand oscillatory motion. As a start, we will say that the oscillatory motion has an angular frequency \( \omega \) that is the same as the angular velocity \( \omega \) of the rotating wheel when the mass and the stick go up and down together.

Figure 2a
Lecture setup for comparing the oscillation of a mass on a spring with a rotating wheel. A stick is mounted on the rotating wheel, and a TV camera off to the side provides a side view of the oscillating mass and rotating wheel.

Figure 2b
Side view of the oscillating mass and rotating wheel, as seen by the TV camera. When the motor is adjusted to the correct frequency, the mass and the stick are observed to move up and down together.
THE SINE WAVE

There is another way to picture the sideways view of circular motion. As more or less a thought experiment, suppose that we take our rotating wheel with a stick shown in Figure (2a) and shine a parallel beam of light at it, sideways, as shown in Figure (3). Now picture a truck with a big billboard mounted on the back, driving away from the light source as shown on the right side of Figure (3). The image of the stick will move up and down on the billboard as the truck moves forward.

Finally picture the line traced out in space by the image of the stick on the moving billboard. The image is going up and down with a frequency $\omega$ and moves forward at a speed $v$, the speed of the truck. The result is an undulating curve we call a sine wave.

To make this definition of the sine wave more specific, assume the truck crosses the point $x = 0$ just when the angle $\theta$ of the stick is zero as shown in Figure (3). Let us imagine that the truck drives at a speed $v = \omega$, so that the distance $x = vt = \omega t$ that the truck has travelled is the same as the angular distance $\theta = \omega t$ that the stick has travelled. Finally let the radius of the circle around which the stick is travelling be $r = 1$, so that the undulating curve goes up to a maximum value $y = +1$ and down to a minimum value $y = -1$. With these conditions, the curve traced out is the mathematical function

$$y = \sin \theta = \sin (\omega t)$$  

Let us remove the truck and billboard and look at the sine curve itself more carefully as shown in Figure (4). The horizontal axis of the sine curve is the angular distance $\theta = \omega t$ that the rotating stick has travelled. Starting at 0 when $\theta = 0$, the sine curve completes one full cycle or undulation just when the wheel has gone around once and $\theta = 2\pi$. Thus one cycle of a sine wave goes from 0 to $2\pi$ as shown in Figure (4). The sine wave reaches a maximum height at $\theta = \pi/2$, goes back to zero again at $\theta = \pi$, has a minimum value at $3\pi/2$ and completes the cycle at $\theta = \omega t = 2\pi$.

![Figure 3](image)

Figure 3
Project the image of the stick onto the back of a truck moving at a speed $v = \omega t$, and the image traces out a sine wave.

![Figure 4](image)

Figure 4
The sine curve $y = \sin \theta = \sin (\omega t)$.
Exercise 1
Somewhere back in the dim past, you learned that $\sin \theta$ was the ratio of the opposite side to the hypotenuse in a right triangle. Applied to Figure (5), this is

$$\sin \theta \equiv \frac{y}{r}$$

(2)

Show that this older definition of $\sin \theta$, at least for angles $\theta$ between 0 and $\pi/2$, is the same as the definition of $\sin \theta$ we are using in Figure (3) and (4).

As you can see from Exercise 1, our rotating stick picture of the sine wave is mathematically equivalent to the definition of $\sin \theta$ you learned in your first trigonometry class. What may be new conceptually is the dynamic aspect of the definition. Figures (3) and (4) connect rotational motion to oscillatory motion and to the shape of a sine wave. The relationship between the static picture and the dynamic one is that the angular distance $\theta$ is equal to the angular velocity $\omega$ times the elapsed time $t$.

The basic question for the dynamic picture is how long does one oscillation take. The time for one oscillation is called the period $T$ of the oscillation. We can therefore ask what is the period $T$ of an oscillation whose angular velocity, or angular frequency is $\omega$.

The solution to this problem is to note that the sine wave completes one cycle when $\theta = 2\pi$. But $\theta$ is just the angular distance $\omega t$. Thus, if $t = $ one period $T$, we have

$$\theta = \omega T = 2\pi$$

T = $ \frac{2\pi}{\omega}$ period of a sinusoidal oscillation

(3)

To remember formulas like Equation 3, we can use the same set of dimensions we used in our discussion of angular motion. If we remember the dimensions

- $\omega$ radians second $\text{angular frequency}$
- $2\pi$ radians cycle
- $T$ seconds cycle $\text{period}$
- $f$ cycles second $\text{frequency}$

then we can go back and forth between the quantities $\omega$, $T$ and $f$ simply by making the dimensions come out right.
For example

\[ T \text{ sec} \text{ cycle} = \frac{2\pi \text{ radians}}{\omega \text{ radians sec}} = \frac{2\pi}{\omega} \text{ sec} \text{ cycle} \]

\[ f \text{ cycles sec} = \frac{\omega \text{ radians sec}}{2\pi \text{ radians cycle}} = \frac{\omega}{2\pi} \text{ cycles sec} \]

\[ T \text{ sec} \text{ cycle} = \frac{1 \text{ cycles sec}}{f \text{ sec cycle}} = \frac{1}{f} \text{ sec} \text{ cycle} \]

We repeated these dimensional exercises, because it is essential that you be able to easily go back and forth between quantities like frequency, angular frequency, and period.

**Exercise 2**

(a) A spring is vibrating at a rate of 2 seconds per cycle. What is the angular velocity \( \omega \) of this oscillation?

(b) What is the period of oscillation, in seconds, of an oscillation where \( \omega = 1 \) ?

---

**Exercise 3**

As shown in Figure (6), an air cart sitting on an air track has springs attached to the ends of the track as shown. We are taking \( x = 0 \) to be the equilibrium position of the cart. It turns out that the car oscillates back and forth with the same kind of sinusoidal motion as the mass on the end of a spring shown in Figures (1) and (2).

Assume that the mathematical formula for the coordinate \( x \) of the cart is

\[ x = x_0 \sin \omega t \]

where

\[ x_0 = 4 \text{ cm} ; \quad \omega = 3 \text{ radians sec} \]

(a) Figure (7) is an \( x \)-\( t \) graph of the position of the air cart in Figure (6). We have drawn in the sine curve, and drawn tick marks at important points along the \( x \) and \( t \) axis. On the \( x \) axis, the tick marks are at the maximum and minimum values of \( x \). On the \( t \) axis, the tick marks are at 1/4, 1/2, 3/4 and 1 complete cycle. Insert on the graph, the numerical values that should be associated with these tick marks.

(b) Where will the cart be located at the time \( t = 300 \pi \) seconds?

---

**Figure 6**

Mass with springs on an air cart. We take the equilibrium position to be \( x = 0 \).

**Figure 7**

The \( x \)-\( t \) graph for the motion of the air cart in Figure 6.
**Phase of an Oscillation**

Our sine waves, by definition, begin at 0 when $\theta = 0$. This is equivalent to saying that in Figure (2), the truck crossed $x = 0$ at the same instant that the rotating wheel crossed $\theta = 0$. We did not have to make this choice, the rotating wheel could have been at any angle $\phi$ when the truck crossed $t = 0$ as shown in Figure (8). If the stick were up at an angle $\phi$ when the truck crossed the zero of the horizontal axis, we say that the resulting sine wave has a phase $\phi$. The formula for the resulting sine curve is

$$y = \sin (\omega t + \phi)$$

You can see from this equation that at $t = 0$, the angle of the sine wave is $\theta = \omega t + \phi = \phi$

![Figure 8](image)

*Figure 8*  
*The phase angle $\phi$.*

In Figure (9) we have sketched the sine wave for several different phases. At a phase $\phi = \pi / 2$, the wave starts at 1 for $\theta = 0$ and goes down to 0 at $\theta = \pi / 2 = 90^\circ$. This is just what $\cos \theta$ does, and we have what is called a cosine wave. We have

$$\cos (\omega t) = \sin (\omega t + \pi / 2)$$

When the phase angle gets up to $\pi$ or $180^\circ$, the sine wave is reversed into a $-\sin (\omega t)$ wave. At $\phi = 2\pi$ we are back to the sine wave we started with.

![Figure 9](image)

*Figure 9*  
*Various phases of the sine wave. When $\phi = \pi / 2$, the wave is called a cosine wave. (It matches the definition of a cosine, which starts out at 1 for $\theta = 0$.)*

*At a phase angle $\phi = \pi$ or $180^\circ$, the sine wave has reversed and become $-\sin (\omega t)$. At $\phi = 2\pi$, we are back to a sine wave again.*
Exercise 4
In trigonometry class, or somewhere perhaps, you were given the trigonometric identity
\[ \sin(a + b) = \sin a \cos b + \sin b \cos a \]
Use this result to show that
\[ \sin(\omega t + \pi/2) = \cos \omega t \quad (7a) \]
\[ \sin(\omega t + \pi) = -\sin \omega t \quad (7b) \]
\[ \sin(\omega t + 3\pi/2) = -\cos \omega t \quad (7c) \]
\[ \sin(\omega t + 2\pi) = \sin \omega t \quad (7d) \]
These are the results graphed in Figure (9).

**MASS ON A SPRING; ANALYTIC SOLUTION**

Let us now apply Newton’s laws to the motion of a mass on a spring and see how well the results compare with the sinusoidal motion we observed in the demonstration of Figure (2). In our analysis of the spring pendulum in Chapter 9 (see Figure 9-4) we saw that the spring exerted a force whose strength was linearly proportional to the amount the spring was stretched, a result known as Hooke’s law.

When we have the one dimensional motion of a mass oscillating about its equilibrium position, as illustrated in Figure (10), then we get a very simple formula for the net force on the object. If we displace the cart of Figure (10) by a distance \( x \) from equilibrium, there is a restoring force whose magnitude is proportional to \( x \) pushing the cart back toward the equilibrium position.

We can completely describe this restoring force by the formula
\[ F = -kx \quad \text{(Hooke’s law)} \quad (8) \]
If \( x = 0 \), the cart is at its equilibrium position and there is no force. If \( x \) is positive, as in Figure (10), the restoring force is negative, pointing back to the equilibrium position. And if \( x \) is negative, the restoring force points in the positive direction. All these cases are handled by the formula \( F = -kx \).

For a mass bobbing up and down on a spring, shown in Figure (11), there is both a gravitational and a spring force acting on the mass. But if you measure the net force starting from the equilibrium position, you still get a linear restoring force. The net force is always directed back toward the equilibrium position and has a strength proportional to the distance \( x \) the mass is away from equilibrium. Thus Equation 8 still describes the net force on the mass.
Exercise 5
Describe experiments you could carry out in the laboratory to measure the force constant \( k \) for the air cart setup of Figure (10). To do this you are given the air cart setup, a string, a pulley, and some small weights.

Sketch the setup you would use to make the measurements, and include some simulated data to show how you would obtain a numerical value of \( k \) from your data. (This is the kind of exercise you would do ahead of time if you planned to do a project studying the oscillatory motion of the air cart and spring system.

To apply Newton’s laws to the problem of the oscillating mass, let \( x(t) \) be the displacement from equilibrium of either the air cart of Figure (10) or the mass of Figure (11). The velocity \( v_x \) and the acceleration \( a_x \) of either the cart or mass is

\[
v_x = \frac{dx(t)}{dt} \quad (9)
\]

\[
a_x = \frac{dv_x}{dt} = \frac{d^2x(t)}{dt^2} \quad (10)
\]

The \( x \) component of Newton’s second law becomes

\[
F_x = ma_x
\]

\[
- kx = m \frac{d^2x}{dt^2} \quad (11a)
\]

where we used Hooke’s law, Equation 8 for \( F_x \).

The result, Equation 11a, involves both the variable \( x(t) \) and its second derivative \( d^2x/dt^2 \). An equation involving derivatives is called a differential equation, and one like Equation 11a, where the highest derivative is the second derivative, is called a second order differential equation. Differential equations are harder to solve than algebraic equations like \( x^2 = 4 \), because the answer is a function or a curve, rather than simple numbers like \( \pm 2 \).

When working with differential equations, there is a traditional form in which to write the equation. The highest derivative is written to the far left, all terms with the unknown variable are put on the left side of the equation, and the coefficient of the highest derivative is set to one. (A reason for this tradition is that only a few differential equations have been solved. If you write them all in a standard form, you may recognize the one you are working with.) Putting Equation 11a in the standard form by dividing through by \( m \) and rearranging terms gives

\[
\frac{d^2x}{dt^2} + \frac{k}{m}x = 0
\]

A standard way to solve a differential equation is to guess the answer, and then plug your guess into the equation to see if the guess works. A course in differential equations basically teaches you how to make good guesses. In the absence of such a course, we have to use whatever knowledge we have about the system in order to make as good a guess as we can. That is why we did the demonstration of Figure (2). In that demonstration we saw the oscillating mass moved the same way as a stick on a rotating wheel, when the wheel is viewed sideways. We then saw that this sideways view of rotating motion is described by the mathematical function \( \sin \omega t \). From this we suspect that a good guess for the function \( x(t) \) may be

\[
x(t) = \sin \omega t \quad \text{initial guess} \quad (12)
\]

In order to see if this guess is any good, we need to substitute values of \( x \) and \( d^2x/dt^2 \) into Equation 11. To do this, we need derivatives of \( x = \sin \omega t \).

From your calculus course you learned that

\[
\frac{d}{dt} \sin \omega t = \omega \cos \omega t \quad (13)
\]

\[
\frac{d}{dt} \cos \omega t = -\omega \sin \omega t \quad (14)
\]
Thus if we start with
\[ x = \sin(\omega t) \]  
(12)
we have
\[ \frac{dx}{dt} = \omega \cos(\omega t) \]  
(15)
\[ \frac{d^2x}{dt^2} = \frac{d}{dt}(\omega \cos(\omega t)) = \omega (-\omega \sin(\omega t)) \]
\[ = \omega (-\omega \sin(\omega t)) \]
(16)
To check our guess \( x = \sin(\omega t) \), we substitute the values of \( x \) from Equation 12 and \( \frac{d^2x}{dt^2} \) from Equation 16 into the differential Equation 11. We get
\[ \frac{d^2x}{dt^2} + kmx = 0 \]
\[ \left[ -\omega^2 \sin(\omega t) \right] + \frac{k}{m} \left[ \sin(\omega t) \right] = 0 \]  
(17)
We put the question mark over the equal sign, because the question we want to answer is whether \( x = \sin(\omega t) \) can be a solution to our differential equation. Can the sum of these two terms be made equal to zero as required by Equation 11?

The first thing we note in Equation 17 is that the function \( \sin(\omega t) \) cancels. This is encouraging, because if we ended up with two different functions of time, say a \( \sin(\omega t) \) in one term and a \( \cos(\omega t) \) in the other, there would be no way to make the sum of the two terms to be zero for all time, and we would not have solved the equation. However because the \( \sin(\omega t) \) cancels, we are left with
\[ -\omega^2 + \frac{k}{m} \neq 0 \]  
(18)

Equation 18 is easily solved with the choice
\[ \omega = \sqrt{\frac{k}{m}} \]
(19)
Thus we have shown not only that \( x = \sin(\omega t) \) is a solution of Newton’s second law, but we have also solved for the frequency of oscillation \( \omega \). Newton’s second law predicts that the air cart will oscillate at a frequency \( \omega = \sqrt{\frac{k}{m}} \). This result is easily tested by experiment.

**Exercise 6**
(a) In the formula \( x = \sin(\omega t) \), \( \omega \) is the angular frequency of oscillation, measured in radians per second. Using the formula \( \omega = \sqrt{\frac{k}{m}} \) and dimensional analysis, find the predicted formula for the period \( T \) of the oscillation, the number of seconds per cycle.

(b) A mass \( m = 245 \) gms is suspended from a spring as shown in Figure (12). The mass is observed to oscillate up and down with a period of 1.37 seconds. From this determine the spring constant \( k \). How could this result have been used in the spring pendulum experiment discussed in Chapter 9 (page 9-3)?

<table>
<thead>
<tr>
<th>Figure 12</th>
</tr>
</thead>
<tbody>
<tr>
<td>The spring constant can be determined by measuring the period of oscillation.</td>
</tr>
</tbody>
</table>

(You can check your answer, since the ball and spring of Figure (12) are the same ones we used in the spring pendulum experiment.)
The guess we made in Equation 12, \( x = \sin (\omega t) \) is not the only possible solution to our differential Equation 11. In the following exercises, you show that
\[
x = A \sin (\omega t)
\]  
(12a)
is also a solution, where \( A \) is an arbitrary constant. Since the function \( \sin (\omega t) \) oscillates back and forth between the values +1 and –1, the function \( A \sin (\omega t) \) oscillates back and forth between +\( A \) and –\( A \). Thus \( A \) represents the amplitude of the oscillation. The fact that Equation 12a, with arbitrary \( A \), is also a solution to Newton’s second law, means that a sine wave with any amplitude is a solution. (This is true as long as you do not stretch the spring too much. If you pull a spring out too far, if you exceed what is called the elastic limit, the spring does not return to its original shape and its spring constant changes.)

**Exercise 7**

As a guess, try Equation 12a as a solution to the differential Equation 11. Follow the same kind of steps we used in checking the guess \( x = \sin (\omega t) \), and see why Equation 12a is a solution for any value of \( A \).

**Exercise 8**

(a) Show that the guess
\[
x = A \cos (\omega t)
\]  
(12b)
is also a solution to our differential Equation 11. This should be an expected result, because the only difference between a sine wave and a cosine wave is the choice of the time \( t = 0 \) when we start measuring the oscillation.

(b) The sine and cosine waves are only special cases of the more general solution
\[
x = A \sin (\omega t + \phi)
\]  
(12c)
where \( \phi \) is the phase of the oscillation discussed in Figure (9). Show that Equation 12c is also a solution of our differential Equation 11. [Hint: the derivative of \( \sin (\omega t + \phi) \) is \( -\omega \cos (\omega t + \phi) \). You can, if you want, prove this result using Equations 13 and 14 and the trigonometric identities
\[
\sin (a + b) = \sin a \cos b + \cos a \sin b
\]
\[
\cos (a + b) = \cos a \cos b - \sin a \sin b
\]
Remember that \( \phi \) is a constant.]

**Exercise 9**

We do not want you to think every function is a solution to Equation 11. Try as a guess
\[
x = e^{-\alpha t}
\]  
(20)
which represents an exponentially decaying curve shown in Figure (13). To do this you need to know that
\[
\frac{d}{dt} e^{-\alpha t} = -\alpha e^{-\alpha t}
\]  
(21)
When you try Equation 20 as a guess, what goes wrong? Why can’t this be a solution to our differential equation? [Or, by what crazy way could you make it a solution?]
Conservation of Energy

Back in Chapter 10 we calculated the formula for the potential energy stored in the springs when we pulled the cart of Figure (10) a distance $x$ from equilibrium. The result was

$$ U_{\text{spring}} = \frac{1}{2} k x^2 \tag{10-28} $$

We then used the law of conservation of energy to predict how fast the cart would be moving when it crossed the $x = 0$ equilibrium line if it were released from rest at a position $x = x_0$. The idea was that the potential energy $\frac{1}{2} k x_0^2$ the springs have when the cart is released, is converted to kinetic energy $\frac{1}{2} m v_0^2$ the cart has when it is at $x = 0$ and its speed is $v = v_0$.

**Exercise 10**

See if you can derive Equation 10-28 without looking back at Chapter 10. If you cannot, review the derivation now.

Using conservation of energy to predict the speed of the air cart was particularly useful back in Chapter 10 because at that time we did not have the analytic solution for the motion of the cart. Now that we have solved Newton’s second law to predict the motion of the cart, we can turn the problem around, and see if energy is conserved by the analytic solution.

An analytic solution for the position $x(t)$ and velocity $v(t)$ of the cart is

$$ x(t) = \sin \left( \omega t \right) \tag{22} $$

$$ v(t) = \frac{dx}{dt} = \omega \cos \left( \omega t \right) $$

For this solution, the kinetic and potential energies are

$$ \text{kinetic energy} \quad \frac{1}{2} m v^2 = \frac{1}{2} m \omega^2 \cos^2(\omega t) \tag{23} $$

$$ \text{potential energy} \quad \frac{1}{2} k x^2 = \frac{1}{2} k \sin^2(\omega t) \tag{24} $$

The total energy $E_{\text{tot}}$ of the cart at any time $t$ is

$$ E_{\text{tot}} = \text{kinetic energy} + \text{potential energy} $$

$$ E_{\text{tot}} = \frac{1}{2} m \omega^2 \cos^2(\omega t) + \frac{1}{2} k \sin^2(\omega t) \tag{25} $$

At first sight, Equation 25 does not look too promising. It seems that $E_{\text{tot}}$ is some rather complex function of time, hardly what we expect if energy is conserved. However remember that the frequency $\omega$ is related to the spring constant $k$ by $\omega = \sqrt{k/m}$, thus we have

$$ \frac{1}{2} m \omega^2 = \frac{1}{2} m \left( \sqrt{\frac{k}{m}} \right)^2 $$

$$ = \frac{1}{2} m \frac{k}{m} = \frac{1}{2} k \tag{26} $$

Thus the two terms in our formula for $E_{\text{tot}}$ have the same coefficient $\frac{1}{2} k$, and $E_{\text{tot}}$ becomes (using Equation 26 in 25)

$$ E_{\text{tot}} = \frac{1}{2} k \left[ \cos^2(\omega t) + \sin^2(\omega t) \right] \tag{27} $$

Equation 27 can be simplified further using the trigonometric identity

$$ \cos^2(a) + \sin^2(a) = 1 \tag{28} $$

for any value of $a$. Thus the term in square brackets in Equation 27 has the value 1, and we are left with

$$ E_{\text{tot}} = \frac{k}{2} \tag{29} $$

The total energy of the mass and spring system is constant as the oscillator moves back and forth. Energy is conserved after all!

**Exercise 11**

What is the total energy of an oscillating mass whose amplitude of oscillation is $A$? [Start with the solution $x(t) = A \sin(\omega t)$, calculate $v(t)$, and then calculate $E_{\text{tot}} = \frac{1}{2} k x^2 + \frac{1}{2} m v^2$.]
14-12 Oscillations and Resonance

THE HARMONIC OSCILLATOR

The sinusoidal motion we have been discussing, which results when an object is subject to a linear restoring force $F = -kx$, is called simple harmonic motion and the oscillating system is often called a harmonic oscillator. These general names are used because there are many examples in physics of simple harmonic motion. In some cases the sine wave solution $\sin(\omega t)$ is an exact solution of a Hooke’s law problem. In many other cases, the solution is approximate, valid only for small amplitude oscillations where the displacements $x$ do not become too big. In the following sections we will consider examples of both kinds of problems.

The Torsion Pendulum

One example of simple harmonic motion is provided by part of the apparatus used by Cavendish to detect the gravitational force between two lead balls. The apparatus, illustrated back in Figure (8-8), contains two small lead balls mounted on a light rod, which in turn is suspended from a glass fiber as shown in Figure (14a). (Such glass fibers are easy to make. Heat the center of a glass rod in a Bunsen burner until the glass is about to melt, and then pull the ends of the rod apart. The soft glass stretches out into a long thin fiber.)

If you let the rod with two balls come to rest at its equilibrium position, then rotate then rod by an angle $\theta$ in the horizontal plane as shown in the top view of Figure (14b), the glass fiber exerts a torque tending to rotate the rod back to its equilibrium position. Careful experiments have shown that the restoring torque exerted by the glass fiber is proportional to the angular distance $\theta$ that the rod has been rotated from equilibrium. The rod is acting like an angular spring, producing a restoring torque $\tau_r$ given by an angular version of Hooke’s law

$$\tau_r = -k\theta$$

(30)

In the Cavendish experiment two large balls of mass $M$ are placed near the small balls as shown in Figure (15).
The gravitational forces $\vec{F}_g$ between the large and small balls produce a net torque $\tau_g$ on the suspended rod of magnitude

$$\tau_g = 2\left( rF_g \right)$$

(31)

where $r$ is the distance from the center of the rod to the small mass (the lever arm), and the factor of 2 is from the fact that we have a gravitational force acting on each pair of the balls.

The suspended rod will finally come to rest at an angle $\theta_0$ where the gravitational torque $\tau_g$ just balances the glass fiber restoring torque $\tau_r$, so that there is no net torque on the rod. Equating the magnitudes of $\tau_r$ and $\tau_g$, Equations 30 and 31 gives us

$$|\tau_g| = |\tau_r|$$

$$2rF_g = k\theta_0$$

(32)

Equation 32 can then be solved for the gravitational force $\tau_g$ in terms of the rod length $2r$, the restoring constant $k$, and the rest angle $\theta_0$.

The problem the Cavendish experiment has to overcome is the fact that the gravitational force between the two lead balls is extremely weak. You need an apparatus where the tiny gravitational torque $\tau_g$ produces an observable deflection $\theta_0$. That means that the restoring torque $\tau_r$ must also be very small. That was why the long glass fiber was used to suspend the rod, for it produces an almost immeasurably small restoring torque.

In order to carry out the experiment and measure the gravitational force $F_g$, you need to know the restoring torque constant $k$ that appears in Equation 32. But the feature of the glass fiber that makes it good for the experiment, the small value of $k$, makes it hard to directly measure the value of $k$. To determine $k$ by direct measurement would mean applying known forces of magnitude $F_g$, but the only forces around that are sufficiently weak are the gravitational forces you are trying to measure.

Fortunately there is an easy way to obtain an accurate value of the restoring constant $k$. Remove the large lead balls, displace the rod from equilibrium by some reasonable angle $\theta$ as shown in Figure (14b), and let go. You will observe the rod to swing back and forth in an oscillatory motion. The rod, two balls, and glass fiber of Figure (14a) form what is called a torsion pendulum, and the oscillation is caused by the restoring torque of the glass fiber. The glass fiber is acting like an angular spring, creating an angular harmonic motion in strict analogy to the linear harmonic motion of a mass suspended from a spring.

The analogy applies directly to the equations of motion of the two systems. For a linear one dimensional system like a mass on a spring, Newton’s second law is

$$F_x = m\frac{d^2x}{dt^2}$$

The angular version of Newton’s second law, applied to the simple case of an object rotating about a fixed axis, is from Equation 30 of Chapter 12

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2}$$

(12-20)

where $\tau$ is the net torque, $I$ the angular mass or moment of inertia, and $\alpha$ the angular acceleration of the object about its axis of rotation.
For the linear harmonic oscillator (mass on a spring), the force \( F_x \) is a linear restoring force \( F_x = -kx \), which gives rise to the equation of motion and differential equation

\[
F_x = -kx = m \frac{d^2x}{dt^2} \tag{11a}
\]

\[
\frac{d^2x}{dt^2} + \frac{k}{m} x = 0 \tag{11}
\]

For our torsion pendulum, the restoring torque is \( \tau_r = -k\theta \) which gives rise to the equation of motion and differential equation

\[
\tau_r = -k\theta = I \frac{d^2\theta}{dt^2} \tag{33a}
\]

\[
\frac{d^2\theta}{dt^2} + \frac{k}{I} \theta = 0 \tag{33}
\]

Equations 11 and 33 are the same if we substitute the angular distance \( \theta \) for the linear distance \( x \), and the angular mass \( I \) for the linear mass \( m \). For the linear motion, we saw that the spring oscillated back and forth at an oscillation frequency \( \omega_0 \) and period \( T \) given by

\[
\omega_0 = \sqrt{\frac{k}{m}}
\]

\[
T = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{m}{k}} \tag{19}
\]

By strict analogy, we expect the torsion pendulum to oscillate with a frequency \( \omega_0 \) and period \( T \) given by

\[
\omega_0 = \sqrt{\frac{k}{I}}
\]

\[
T = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{I}{k}} \tag{34}
\]

where \( I \) is the moment of inertia of the rod and two balls about the axis defined by the glass fiber (as shown in Figure 14).

As a result, by observing the period of oscillation of the rod and two balls (with the big masses \( M \) removed), you can determine the restoring constant \( k \) of the glass fiber, and use that result in Equation 32 to solve for the gravitational force \( F_g \). Because you can measure periods accurately by timing many swings, \( k \) can be measured accurately, and the Cavendish experiment allows you to do a reasonably good job of measuring the gravitational force \( F_g \).

**Exercise 12**

Solve the differential Equation 33 by starting with the guess

\[
\theta(t) = A \sin (\omega_0 t) \tag{35}
\]

Check that Equation 35 is in fact a solution of Equation 33, and find the formula for the frequency \( \omega_0 \) of the oscillation. Also use dimensional analysis find the period of oscillation.

**Exercise 13**

In the commercial Cavendish experiment apparatus shown in Figure 8-8, the small balls each have a mass of 170 gms, the distance between the small balls is 12 cm, and the observed period of oscillation is 24 minutes.

(a) Calculate the value of the restoring constant \( k \) of the glass fiber.

(b) How big a torque, measured in dyne centimeters, is required to rotate the glass fiber by an angle of one degree. (Remember to convert degrees to radians.)
**The Simple Pendulum**

Perhaps the most well-known example of oscillatory motion is the simple pendulum which consists of a mass swinging back and forth on the end of a string or rod. The regular swings of this pendulum serve as the basic timing device of the grandfather’s clock.

When we begin to analyze the simple pendulum, we will find that it is not quite so simple after all. The restoring force is not strictly a linear restoring force and we end up with a differential equation whose solution is more complex than the sinusoidal oscillations we have been discussing. What allows us to include this example in our discussion of simple harmonic motion is the fact that, for small amplitude oscillations, the restoring force is approximately linear, and the resulting motion is approximately sinusoidal.

Figure (16) is a sketch of a simple pendulum consisting of a small mass $m$ swinging on the end of a string of length $\ell$. The downward gravitational force $mg$ has a component of magnitude $(mg \sin \theta)$ directed along the circular path of the ball.

Since the ball is constrained to move along the circular arc, we can analyze the motion of the ball by equating the tangential forces acting along the arc to the mass times the tangential acceleration. The tangential component of the gravitational force is always directed toward the bottom equilibrium position, thus it is a restoring force of the form

$$F_{\text{tangential}} = -mg \sin \theta$$  \hspace{1cm} (36)

As the mass moves along the arc, the speed of the ball is related to the angle $\theta$ by

$$v_{\text{tangential}} = \ell \frac{d\theta}{dt}$$  \hspace{1cm} (37)

a result from the beginning of our discussion of circular motion in Chapter 12. (See the discussion before Equation 12-11.) Differentiating Equation 37, we get for the tangential acceleration

$$a_{\text{tangential}} = \frac{dv_{\text{tangential}}}{dt} = \ell \frac{d^2\theta}{dt^2}$$  \hspace{1cm} (38)

Thus Newton’s second law gives

$$F_{\text{tangential}} = ma_{\text{tangential}}$$

$$-mg \sin \theta = m\ell \frac{d^2\theta}{dt^2}$$  \hspace{1cm} (39)

Dividing through by $m\ell$ and rearranging terms gives us the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell} \sin \theta = 0$$  \hspace{1cm} \text{equation for a simple pendulum}  \hspace{1cm} (40)

Equation 40, the differential equation for the simple pendulum, is more complex than the equations we have been discussing that lead to simple harmonic motion. If you try as a guess that the motion is sinusoidal and try the solution $\theta = \sin (\omega t)$, it does not work. You are asked to see why in the following exercise.

---

**Exercise 14**

Try substituting the guess

$$\theta = \sin(\omega t)$$

into Equation 40 and see what goes wrong. Why can’t you make the left side zero with this guess?
There is a solution to Equation 40, it is just not the sine curve we have been discussing. The solution is a curve called an elliptic integral, a curve generated much as we generated the sine curve in Figure (3), except that the stick whose shadow generates the curve has to move around an elliptical path rather than around the circular path used in Figure (3). Elliptic integrals carry us farther into the theory of functions than we want to go in this text, thus we will not discuss the exact solution of the differential Equation 40.

**Small Oscillations**

The problem with Equation 40 is the appearance of the function \( \sin \theta \) in the second term on the left hand side. It is this term that seems to keep us from using the oscillatory solution.

In Figure (17) we look again at the geometry of the simple pendulum. In that figure we have a right triangle whose small angle is \( \theta \), hypotenuse the string length \( \ell \), and opposite side \( x \). The definition of the sine of the angle \( \theta \) is

\[
\sin \theta \equiv \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{x}{\ell} \tag{41}
\]

The definition of the angle \( \theta \), in radian measure, is the arc length divided by the radius \( \ell \) of the circular arc

\[
\theta \equiv \frac{\text{arc length}}{\text{radius}} = \frac{\text{arc length}}{\ell} \tag{42}
\]

From Figure (17) we see that for small angles \( \theta \) the opposite side \( x \) and the arc length \( \ell \) are about the same. The smaller the angle \( \theta \), the more nearly equal they are. If we restrict our analysis to small amplitude swings, we can replace \( \sin \theta \) by \( \theta \) in Equation 40, giving us the differential equation

\[
\frac{d^2 \theta}{dt^2} + \frac{g}{\ell} \theta = 0 \tag{43}
\]

Equation 43 is an equation for simple harmonic motion. If we try the guess \( \theta = \sin \left( \omega_0 t \right) \), and plug the guess into Equation 43, we can solve the equation provided the frequency \( \omega_0 \) and the period of oscillation \( T \) have the values

\[
\omega_0 = \sqrt{\frac{g}{\ell}} \tag{44}
\]

\[
T = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{\ell}{g}} \tag{44}
\]

**Exercise 15**

Substitute the guess \( \theta = A \sin \left( \omega_0 t \right) \) into Equation 43 and show that you get a solution provided \( \omega_0^2 = g/\ell \). Then use dimensional analysis to derive a formula for the period of the oscillation.

From Equation 44, we see that the period of the oscillation of a simple pendulum depends only on the gravitational acceleration \( g \) and the length \( \ell \) of the pendulum. It does not depend on the mass \( m \) of the swinging object, nor on the amplitude of the oscillation, provided that the amplitude is kept small. For these reasons the simple pendulum makes a good timing device.

**Exercise 16**

How long should a simple pendulum be so that its period of oscillation is one second?
**Simple and Conical Pendulums**

In Chapter 9 we analyzed the motion of a conical pendulum. The conical pendulum also consists of a mass on a string, but the mass is swung around in a circle as shown in Figure (18), rather than back and forth along an arc as for a simple pendulum.

From our analysis of the conical pendulum, we found that the period of rotation was given by the formula

\[ T_{cp} = 2\pi \sqrt{\frac{h}{g}} \text{ period of a conical pendulum} \]  

(9-34)

where \( h \) is the height shown in Figure (18). Considering the trouble we went through to get an approximate solution to the simple pendulum, it seems surprising that Equation 9-34 is an exact solution to Newton’s second law for any achievable radius \( x \) of the circle.

For small circles, where \( x \ll \ell \), the height \( h \) and the string length \( \ell \) are approximately the same and we have

\[ T_{cp} \approx 2\pi \sqrt{\frac{g}{h}} \approx 2\pi \sqrt{\frac{g}{\ell}} \]  

(45)

But this is just the period of a simple pendulum if the oscillations are kept small. Since the two pendulums have the same period for small oscillations, it makes no difference, as far as the period is concerned, whether we swing the balls back and forth or around in a circle. This prediction is easily checked by experiment.

---

**Exercise 17**

You can do your own experiments to show that as you increase the amplitude of a simple pendulum, the period of oscillation starts to get longer. In contrast, when you increase the radius of the circle for a conical pendulum, the height \( h \) and the period \( T_{cp} \) become shorter.

(a) From your own experiments estimate how much longer the period of a simple pendulum is when the maximum angle \( \theta_{max} \) is 90° than when \( \theta_{max} \) is small. (Is it 20% longer, 30% longer? Do the experiment and find out. Does this percentage depend on the length \( \ell \) of the string?)

(b) For a conical pendulum, at what angle \( \theta_{0} \) (shown in Figure 18) is the period half as long as it is for small angles \( \theta_{0} \)? Give your answer for \( \theta_{0} \) in degrees.

---

**Figure 18**

*The conical pendulum.*
Exercise 18
Another way to analyze the simple pendulum is to treat the mass and the string as a rigid object that can rotate about an axis through the top end of the string as shown in Figure (19). Then use the angular version of Newton's second law in the form

$$\tau = | I | \frac{d^2 \theta}{dt^2} \quad (12-20)$$

where $\tau$ is the net torque, and $| I |$ the moment of inertia, of the mass and string about the axis 0.

(a) When the string is at an $\theta$ angle as shown in Figure (19), what is the torque $\tau$ about the axis 0, exerted by the gravitational force $mg$?

(b) What is the moment of inertia $| I |$ of the mass and string about the axis 0?

(c) Show that when you use the above values for $\tau$ and $| I |$ in Equation 12-20 you get the same differential Equation 40 that we got earlier for the simple pendulum.

Exercise 19 A Physical Pendulum
A uniform rod of length $l$ is pivoted at one end as shown in Figure (20). It is free to swing back and forth about this axis, forming what is called a physical pendulum. A simple pendulum is one where the mass is all concentrated at the end as in Figure (19). In a physical pendulum the mass is distributed in some other way, in this case uniformly along the rod.

(a) What is the torque $\tau$ about the axis 0 exerted by the gravitational force on the rod? (In Chapter 13 near Equation 13-11, we showed that when calculating the torque exerted by a gravitational force, you may assume that all the mass is concentrated at the center of gravity of the object.)

(b) What is the moment of inertia $| I |$ of the rod about the axis at the end of the rod? (See exercise 5 in Chapter 12.)

(c) Write the differential equation for the motion of the rod. (Use the procedure outlined in Exercise 18.)

(d) Find the period of small oscillations of the rod.
NON LINEAR RESTORING FORCES

The simple pendulum is an example of an oscillator with a non linear restoring force. In Figure (21), we show the actual restoring force \( (mg \sin \theta) \) and the linear approximation \( (mg \theta) \) that we used in order to solve the differential equation for the pendulum’s motion. You can see that if the angle \( \theta \) always remains small, much less than \( \pi/2 \) in magnitude, then the linear force \( (mg \theta) \) is a good approximation to the non linear force \( (mg \sin \theta) \). Since the linear force gives rise to sinusoidal simple harmonic motion, we expect sinusoidal motion for small oscillations of the simple pendulum. What we are seeing is that a linear restoring force is described by a straight line, and that the non linear restoring force can be approximated by a straight line in the region of small oscillations.

Figure 21
The non linear restoring force \( mg \sin \theta \) can be approximated by the straight line (linear term) \( mg \theta \) if we keep the angle \( \theta \) small.
MoLecular Forces
One of the most important examples of a non-linear restoring force is the molecular force between atoms. Consider, for example, the hydrogen molecule which consists of two hydrogen atoms held together by a molecular force. (We will discuss the origin of the molecular force in Chapter 17.)

In the hydrogen molecule, the hydrogen atoms have an equilibrium separation, and the molecular force provides a restoring force to this equilibrium separation. The restoring force, however, is quite non-linear. If you try to squeeze the atoms together, you quickly build up a large repulsive force that keeps the atoms from penetrating far into each other.

If you try to pull the atoms apart, there is an attractive force that pulls the atoms back together. The attractive force never gets too big, and then dies out when the separation gets much larger than an atomic diameter.

In Figure (22) we have sketched the molecular force as a function of the separation of the atoms, the origin being at the equilibrium position. This graph is not too unlike Figure (21) where we have the force curve for the simple pendulum. For the pendulum, the equilibrium position is at $\theta = 0$, thus the origin of both curves represents the equilibrium position.

While the overall shape of the force curves for the simple pendulum and the molecular force are quite different, right in close to the origin both curves can be approximated by a straight line, a linear restoring force. As long as the amplitudes of the oscillation remain small, we effectively have a linear restoring force and any oscillations should be simple harmonic motion.

In Chemistry texts one often sees molecular forces as being represented by springs as shown in Figure (23). The spring force, given by Hooke’s law, is our ideal example of a linear restoring force. We can now see that, while the molecular force in Figure (22) does not look like a linear spring force, if the amplitude of oscillation remains small, the spring force provides a reasonably good approximation to the actual molecular force. The chemist’s diagrams are not so bad after all.

In a crystal, like quartz, where you have many atoms held together by molecular forces, it is possible to get all the atoms oscillating together. Each atom only oscillates a very small distance about its equilibrium position, but all the oscillations can add up to produce a fairly large, quite detectable oscillation of the crystal as a whole. An advantage of a quartz crystal is that these oscillations can be both driven and detected by electric fields. This vibration or simple harmonic motion of a small quartz crystal is used as the basic timing device for digital watches, computers, and almost all forms of modern electronics.

In Galileo’s time we used small oscillations of a non-linear harmonic oscillator, the simple pendulum, as a basic time device. Now we use the small oscillations of a non-linear harmonic oscillator, the atoms in a quartz crystal, as our most convenient timing device. The main thing we have changed in the last 300 years is not the basic physics, but the size and frequency of the device.

Figure 22
Sketch of the molecular force between two hydrogen atoms. As long as the atoms stay close to the equilibrium position, the force can be represented by a straight line—a linear restoring force.

Figure 23
Representation of the molecular force by a spring force.
DAMPED HARMONIC MOTION

If you start a pendulum swinging or a quartz crystal oscillating, and do not keep the oscillation going with some kind of external force, the oscillation will eventually die out due to friction forces. Such a dying oscillation is called **damped harmonic motion**.

The analysis of damped harmonic motion starts out quite easily. In Newton’s second law add a damping force like the air resistance term we added to our analysis of projectile motion. (See Chapter 3, Figure 31.) We could write, for example

\[ F_{\text{tot}} = F_{\text{restoring}} + F_{\text{damping}} \]

\[ F_{\text{tot}} = -kx - b v_x \]  \hspace{1cm} (46)

where \( x(t) \) is the coordinate of the oscillator, \( v_x(t) = \frac{dx}{dt} \) its velocity, and we are assuming a simple linear damping proportional to \(-v\) with a strength \( b \).

Using Equation 46 in Newton’s second law gives

\[ F_{\text{tot}} = -kx - bv_x = ma_x \]  \hspace{1cm} (47)

With \( a_x = \frac{d^2x}{dt^2} \), this becomes, after dividing through by \( m \) and rearranging terms

\[ \frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = 0 \]  \hspace{1cm} (48)

Equation 48 is our new differential equation for damped harmonic motion. It is like our old differential Equation 11a for undamped oscillation, except that it has the additional term \( \frac{b}{m} \frac{dx}{dt} \) representing the damping.

If we were doing a computer solution of harmonic motion, adding the damping term represents hardly any extra effort at all. In the appendix to this chapter we discuss a short computer program to handle harmonic motion. Starting with the first version of the program that has no damping, you can include damping by changing the line

\[ \text{LET } F = -k \ast x \]

to the new line

\[ \text{LET } F = -k \ast x - b \ast v \]  \hspace{1cm} (49)

and placing your choice for the damping constant \( b \) in the initial conditions.

In contrast, when working with differential equations analytically you find that a very small change in the equation can make a great deal of difference in the effort required to obtain a solution. Adding a bit of damping to a harmonic oscillator changes the curve from a pure sinusoidal motion to a dying sine wave. If you try using a pure sine wave as a guess for the solution to the differential Equation 48 for damped harmonic motion, the guess does not work because the pure sine wave has the wrong shape. The decay of the sine wave has to be built into your guess before the guess stands a chance of working.

The difficult part about solving differential equations is that you essentially have to know the answer before you can solve the equation. You only have to know general features like the fact that in working with Equation 11a you are dealing with a sinusoidal oscillation. You can then use the differential equation to determine explicit features like the frequency of the oscillation. It is helpful to have a physical example to tell you what the general features of the motion are, so that you can begin the process of solving the equation. That is why we begin this chapter with the demonstration in Figure (2) that the motion of a mass on a spring is similar to circular motion seen sideways, namely sinusoidal motion.
To set up a physical model for damped harmonic motion is not too difficult. One way to add damping to the air cart and springs oscillator is to run a string from the air cart over a pulley to a small weight as shown in Figure (24). The idea was to have the weight move up and down in a glass of water to give us fluid damping. But it turned out that there was enough friction in the pulley itself to give us considerable damping.

To record the motion of the cart, we used the air cart velocity detector that we used in Chapter 8 to study the momentum of air carts during collisions. Figure (25a) shows the velocity of the air cart damped only by the friction in the pulley. In Figure (25b) water was added to the glass so that the weight on the string was moving up and down in water. The result was considerably more damping with the curve almost dying out before any oscillations take place.

It turns out that mechanical oscillators like a pendulum or a mass on a spring are not particularly convenient devices for studying damped harmonic motion, or forced harmonic motion which is the subject of the next section. It is hard to control the damping, just adding the pulley in Figure (24) gave us almost too much damping. Worse yet, the damping that we get from friction in a pulley, or a mass moving up and down in water, is not a simple linear damping force of the form $-bv$. What is remarkable about these systems is that much more complex forms of damping give us results similar to what we would get with linear damping.

In Chapter 27 we will study the behavior of basic electric circuits made from electrical components called capacitors, inductors, and resistors. It turns out that the amplitude of the currents in these circuits obey differential equations that are exactly like our oscillator Equations (11) and (45). The damping is caused by the resistor in the circuit, the damping is accurately given by a linear damping term proportional to the amount of resistance in the circuit. (The resistance can be changed simply by turning a knob on a resistance pot.)

Figure (26), taken from Chapter 27, is an example of damped harmonic motion in an electric circuit. Here we have a curve with enough oscillations so that we can see how the wave is damped. In Chapter 27 we will see that the amplitude of the oscillation dies exponentially, following a mathematical curve of the form $e^{-\alpha t}$. As a result, the wave in Figure (26) has the form

$$x = A e^{-\alpha t} \sin \omega t$$

(50)

**Figure 24**
Adding damping to the air cart oscillator.

**Figure 25a**
Damping caused by the pulley and weight alone.

**Figure 25b**
Resulting motion when water was added to the glass.
It turns out that if we use Equation 50 as our guess for a solution to our differential Equation 48 for damped harmonic motion, the guess works, and we can determine both the frequency $\omega$ and decay rate $\alpha$ in terms of the constants that appear in Equation 48.

When we study electric circuits, you will get much more experience with the exponential function $e^{-\alpha t}$, and you will have a better laboratory setup for studying damped and forced harmonic motion. In other words, now, with our somewhat crude mechanical experiments and lack of familiarity with exponential damping, is not the best part of the course to go deeply into the mathematical analysis of these motions. What we will do instead is discuss the motions more or less qualitatively and leave the more detailed analysis for later.

**Exercise 20 Damped Harmonic Motion**

We won’t let you completely off the hook for doing mathematical analysis of damped harmonic motion. Start with Equation 5a as a guess for the form for the displacement $x(t)$ for a damped harmonic oscillator

$$x(t) = Ae^{-\alpha t} \sin(\omega t)$$

![Figure 26](image)

*Figure 26*  
Damped harmonic motion seen in an electric circuit. Note the difference in time scales. The electrical oscillations we will study are usually of much higher frequency than the mechanical ones.

use the following rules of differentiation to calculate $dx/dt$ and $d^2x/dt^2$

$$\frac{d}{dt}e^{-\alpha t} = -\alpha e^{-\alpha t}$$

$$\frac{d}{dt}[a(t)b(t)] = \frac{da}{dt}b + a \frac{db}{dt}$$

and show that when you try this guess in the differential Equation 48, you do in fact get a solution, and that $\omega$ and $\alpha$ are given by

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

$$\alpha = \frac{b}{2m}$$

(51)

You can see that in the absence of damping, when $b = 0$, we get back to our old result $\omega = \sqrt{k/m}$.

**Critical Damping**

In Figure (25b) the damping was so great that the motion damped out almost before the curve had a chance to oscillate. It turns out that there is a critical amount of damping that just kills all oscillations. Any further increase in damping and the mass just coasts to rest.

The idea of critical damping can be seen in our analytic solution for damped harmonic motion obtained Exercise 20. Equation 51 gives us a formula for the frequency of oscillation $\omega$ in terms of the constants $k$ and $b$. We can see as the frequency of oscillation goes to zero, i.e., the period of oscillation becomes infinite when

$$\frac{k}{m} = \frac{b^2}{4m^2}; \quad b = 2\sqrt{mk}$$

$$b = \sqrt{4mk} \quad \text{critical damping}$$

(52)

Equation 52 is the condition for critical damping because if the period of oscillation is infinite there are no oscillations.
RESONANCE

When you are pushing a child on a swing, you time your pushes to coincide with the motion of the child. Usually you give a shove just after the child has swung back and is starting forward again.

If you push forward just as the child is swinging forward, your force $\vec{F}$ and the child’s velocity $\vec{v}$ are in the same direction, the dot product $\vec{F} \cdot \vec{v}$ is positive, and you are adding energy to the child’s motion. Initially the energy you add goes into increasing the amplitude of the swing. After a while friction effects become large enough that the energy you add in each push is dissipated by friction in each swing. (If there is not enough friction, or you push too hard, the child will end up going over the top.)

The key to getting the child swinging was to time your shoves so that $\vec{F} \cdot \vec{v}$ was always positive. If you pushed the child at random intervals, so that $\vec{F} \cdot \vec{v}$ was sometimes positive, sometimes negative, you would be sometimes adding energy and sometimes removing it. The net result would be that your shoves would not be particularly effective in helping the child to swing.

To make sure that you are always adding energy to the child’s swing, you want to time your shoves with the natural frequency of oscillation of the child. When you do this, we say that your shoves are in resonance with the oscillation of the child.

The striking feature of resonance is that a small repeated force can produce a large oscillation. If the damping is small then by adding just a little energy with each shove, the energy accumulates until you end up with a very energetic oscillation. A rather dramatic consequence of this effect is shown in Figure (27) where we see the Tacoma Narrows bridge oscillating wildly and then collapsing.

The new bridge was dedicated in April of 1940. Three months later a reasonably stiff breeze started the bridge oscillating, an oscillation that finally destroyed the bridge’s integrity.

The brute force of the wind itself did not destroy the bridge. The bridge was designed to handle far stronger winds. What happened was that as the wind was blowing over the bridge, vortices began to peel off the bridge. Whenever fluid flows past a cylindrical object at the right speed, vortices began to peel off, first on one side of the cylinder, then the other, and are carried downstream, forming a wake of vortices seen in the wind tunnel photograph of Figure (28). This vortex structure is called a Karmen vortex street after the hydrodynamicist Theodore Von Karmen.

In the case of the Tacoma Narrows bridge, vortices alternately peeled off the top and bottom of the downwind side of the bridge, rocking the bridge at its natural frequency of oscillation. While no separate jolt by any one vortex would have much effect on the bridge, the

**Figure 27a** (See movie on CD)
*Tacoma Narrows bridge oscillating in the winds of a mild gale on Nov 7, 1940.*

**Figure 27b**
*After a couple of hours the bridge collapsed.*
resonance between the peeling of vortices and the oscillation of the bridge caused the oscillations to grow to destructive proportions.

The example of the Tacoma Narrows bridge illustrates how widely the ideas of simple harmonic motion and resonance apply to physical systems. The bridge is far more complex than a mass on a spring, and the vortex street (line of vortices) exerts a rather complex driving force. However the bridge had a natural frequency, the vortices provided a small driving force at that frequency, and we got a resonant amplification of the oscillation.

To apply Newton’s second law to resonant motion, we have to add an oscillating driving force to the system under study. As we have seen from the Tacoma Narrows bridge discussion, we do not need to know the exact form of the driving force, all we need is a repetitive force that can be timed with the natural oscillation. For the theoretical analysis we can use the simplest mathematical form we can find for the driving force, which turns out to be a sine wave.

To write a formula for the driving force, let \( \omega_0 \) be the natural frequency of oscillation (\( \omega_0 = \sqrt{k/m} \) if the damping is small), and let \( \omega \) be the frequency of the driving force. Then the total force, acting on the oscillating system like a mass on a spring, can be written

\[
F_{x\text{tot}} = -kx - bv + F_d \sin(\omega t) \quad (52)
\]

where in the driving term, \( F_d \), represents the amplitude or strength of the sinusoidal driving force. Using Equation 52 in Newton’s second law \( Fx_{\text{tot}} = ma_x \) gives

\[
-kx - b \frac{dx}{dt} + F_d \sin(\omega t) = m \frac{d^2x}{dt^2} \quad (53)
\]

Dividing through by \( m \) and rearranging terms gives

\[
\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = F_d \sin(\omega t) \quad (54)
\]

Equation 54 is the standard form for the differential equation representing forced or resonant harmonic motion. It is the simplest equation we can write whose solution has the features we associate with the phenomena of resonance.

In our study of electric circuits, we can easily create a circuit whose behavior is accurately described by Equation 54. We saw that we could use resistors to add linear damping of the form \( -bv \). It is not hard to add a purely sinusoidal driving force of the form \( F_d \sin(\omega t) \), where we can adjust the driving frequency by turning a knob. In other words, with electric circuits we can accurately study the predictions of Equation 54.

With mechanical systems like a mass on a spring, it is hard to get linear damping, and the sinusoidal driving force is usually simulated by some trick such as wiggling the supported end of the spring at a frequency \( \omega \). Despite the crudeness of the experiment, the equation gives a surprisingly good prediction of what we see.

Since we will later have a laboratory setup that accurately matches Equation 54, we will postpone (until Appendix 1) the mathematical solution of Equation 54. Instead we will investigate the resonance phenomena qualitatively, using the simple setup of a mass on a spring, where we hold the other end of the spring in our hand and move our hand up and down at a frequency \( \omega \) as shown in Figure (29).

**Figure 28**

Karman vortex street in the flow of water past a circular cylinder. The vortices peel off of alternate sides of the cylinder and flow downstream forming a double line of vortices. (Reynolds number = 140.) Photograph by Sadatoshi Taneda.
Resonance Phenomena

We have already seen that if we hold our hand still, pull the mass down, and let go, the mass oscillates up and down at the natural frequency \( \omega_0 = \sqrt{k/m} \). You can observe some damping, because the mass finally stops oscillating. But the damping is small and has no noticeable effect on the resonant frequency. (We can neglect the term \( b^2/4m^2 \) in Equation 51.)

Now try a very different experiment. Stop the mass from oscillating, and slowly move your hand up and down a small distance. If you do this slowly enough and carefully enough, the mass will move up and down with your hand (just as if the spring were not there). In this case the formula for the motion of the mass is

\[
x = x_0 \sin (\omega t) \quad \omega \ll \omega_0
\]

(55)

where the frequency \( \omega \) of the oscillation of the mass is the frequency of \( \omega \) the oscillation of your hand. This only happens if you oscillate your hand at a frequency much much lower than the natural oscillation frequency \( \omega_0 \).

In the next experiment, keep the oscillations of your hand small in amplitude, but start moving your hand up and down rapidly, at a frequency \( \omega \) considerably greater than the natural frequency \( \omega_0 \). Now, what happens is that the mass oscillates at the same frequency as your hand, but out of phase. When your hand is going down, the mass is coming up, and vice versa. Now the formula for the displacement \( x \) of the mass is

\[
x = -x_0 \sin (\omega t) \quad \omega > \omega_0
\]

(56)

where the minus sign tells us that the mass is oscillating out of phase with our hand.

A way we can write both Equations 55 and 56 is in the form

\[
x = x_0 \sin (\omega t + \phi)
\]

(57)

where \( \phi \) is the phase angle of the oscillation (see Figures 8 and 9 and Equation 4 at the beginning of this chapter for a discussion of phase angle.) In Equation 56, where \( \omega \ll \omega_0 \) and there was no phase difference, the phase angle \( \phi \) is zero. In Equation 56 where \( \omega >> \omega_0 \), and the motion is completely out of phase, the phase angle \( \phi \) is \( \pi \) or 180°.

Equations 55 and 56 represent the two extremes of driven harmonic motion. The mass moves with a small amplitude at the same frequency \( \omega \) as the driving force. When the driving frequency is much less than the natural frequency \( \omega_0 \), the difference in phase between the driving force and the response of the mass is zero degrees. When \( \omega >> \omega_0 \) the phase difference increases to \( \pi \) or 180°.

As the third experiment, start at the low frequency where the mass is following your hand, and slowly increase the frequency \( \omega \) of oscillation of your hand, keeping the amplitude of oscillation constant. As \( \omega \) approaches \( \omega_0 \), the amplitude of oscillation of the mass increases. When you get close to the natural frequency \( \omega_0 \), the oscillation becomes so large that the mass will most likely jump off the spring. This is the phenomena of resonance, the phenomenon that destroyed the Tacoma Narrows bridge.

How big the oscillation of the mass becomes depends mainly how close you are to resonance, how close \( \omega \) is to \( \omega_0 \), and how big the damping force is relative to the driving force. The formula for the amplitude \( x_0 \) of the motion of the mass obtained by solving Equation 54 is

\[
x_0 = \frac{F_d/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + \omega^2 b^2 / m^2}}
\]

(58)

where \( F_d \) is the strength of the driving force, \( \omega \) the driving frequency, \( \omega_0 = \sqrt{k/m} \) the natural frequency and \( b \) the damping constant. In the absence of damping (\( b = 0 \)), Equation 58 predicts an infinite amplitude at the resonant frequency \( \omega = \omega_0 \). Such an infinite amplitude is prevented either by damping or by the destruction of the system. (A damping mechanism could have saved the Tacoma Narrows bridge.)
Exercise 21

To derive Equation 21, you start with Equation 57 as a guess

$$x = x_0 \sin(\omega t + \phi)$$  \hspace{1cm} (57)

and substitute that into the differential Equation 54. It turns out that you can get a solution provided that the amplitude $x_0$ has the value given by Equation 58, and the phase angle $\phi$ is given by

$$\tan(\phi) = \frac{b/m}{\omega^2 - \omega_0^2}$$  \hspace{1cm} (59)

Doing the work, actually substituting Equation 57 into 54 and getting Equations 58 and 59 for $x_0$ and $\phi$ is a somewhat messy job which we leave to Appendix 1 on the next page. Here we would rather have you develop an intuitive feeling for the solutions in Equations 58 and 59 by working the following exercises.

(a) Write the formula for $x_0$ in the case $b = 0$. Sketch the resulting curve using the axes shown in Figure (30). Explain what happens as $\omega \to \omega_0$.

(b) When the damping is not zero, find the formula for the amplitude of oscillation $x_0$ at the resonant frequency $\omega \to \omega_0$. Check the dimensions of your answer.

(c) What is the phase angle $\phi$ at resonance? How does the phase angle change as we go from $\omega << \omega_0$ to $\omega >> \omega_0$?

In Figure (31) we have graphed the amplitude $x_0$ for a fixed driving force $F_d$, as a function of $\omega$ for several values of the damping constant $b$. The main point to get from this diagram is that the smaller the damping, the sharper the resonance.

Transients

There is one more qualitative experiment we want to do with our simple apparatus of the hand held spring and mass of Figure (29). Instead of gently starting the mass moving as we had you do in the earlier experiments, let the mass fall from some small height and move your hand up and down at the same time.

If you just let the mass drop from some small height, it will oscillate up and down at the resonant frequency $\omega_0$. If you just start moving your hand slowly at a frequency $\omega$, the mass will move at the same frequency as your hand, building up to an amplitude given more or less by Equation 58 and shown in Figure (31), the driven oscillation we have been discussing.

If you drop the weight and move your hand at the same time, you get both kinds of motion at once. You get the natural oscillation at a frequency $\omega_0$ that eventually dies out due to damping, and the driven oscillation at the frequency $\omega$ that eventually builds up to an amplitude $x_0$. For a while, before the natural oscillation has died out, the resulting motion is a mixture of two frequencies of oscillation and can look quite complex. The natural oscillation is called a transient because it eventually dies out. But until the transients do die out, forced harmonic motion can be fairly complicated to analyze. In the next chapter we will study a powerful technique called *Fourier analysis* that allows us to study complex motions that involve such a mixture of oscillations.

**Figure 30**

*Use these axes to sketch the amplitude vs. frequency for no damping.*

**Figure 31**

*Amplitude of the oscillation for various values of the damping constant. The amplitude of the driving force $F_d$ is the same for all curves.*
APPENDIX 14–1
SOLUTION OF THE DIFFERENTIAL EQUATION FOR FORCED HARMONIC MOTION

The equation we wish to solve is Equation 54
\[
\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = \frac{F_d}{m} \sin(\omega t) \tag{54}
\]
where our guess for a solution is Equation 57
\[
x = x_0 \sin(\omega t + \phi) \tag{57}
\]
The quantities \(x_0\) and \(\phi\) are the unknown amplitude and phase of the oscillation that we wish to determine.

To simply plug our guess into Equation 54 and grind away leads to a sufficiently big mess that we could easily make a mistake. We will instead simplify things as much as possible to make the calculation easier. The first step is to define the constants
\[
\omega_0^2 = \frac{k}{m}; \quad b' = \frac{b}{m}; \quad F' = \frac{F_d}{m} \tag{60}
\]
Next we wish to get the phase angle into the forcing term so that it appears only once in our equation. We can do this by using a time scale \(t'\) where
\[
\omega t' = \omega t + \phi \Rightarrow \omega t = \omega t' - \phi \tag{61}
\]
In terms of the new constants and \(t'\) our differential equation becomes
\[
\frac{d^2x}{dt'^2} + b' \frac{dx}{dt'} + \omega_0^2 x = F' \sin(\omega t' - \phi) \tag{62}
\]
Our guess, and its first and second derivative are
\[
x = x_0 \sin(\omega t') \tag{63a}
\]
\[
\frac{dx}{dt} = \omega x_0 \cos(\omega t') \tag{63b}
\]
\[
\frac{d^2x}{dt'^2} = -\omega^2 x_0 \sin(\omega t') \tag{63c}
\]
In deriving \(dx/dt\) we used the fact that
\[
\frac{dx}{dt} = \frac{dx}{dt'} \frac{dt'}{dt} = \frac{dx}{dt'}
\]
where \(t' = t - \phi/\omega\) so that \(dt'/dt = 1\). We will also use the trigonometric identity
\[
\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b)
\]
to write
\[
F' \sin(\omega t' - \phi) = F' \sin(\omega t' \cos(-\phi) + F' \sin(-\phi) \cos(\omega t') \tag{64}
\]
\[
= F' \sin(\omega t' \cos(\phi) - F' \sin(\phi) \cos(\omega t')
\]
where we used
\[
\cos(-\phi) = \cos(\phi), \quad \sin(-\phi) = -\sin(\phi)
\]
Substituting Equation 63 and 64 into 62, and separately collecting terms with \(\sin(\omega t')\) and \(\cos(\omega t')\), we get
\[
\sin(\omega t') [ -\omega^2 x_0 + \omega_0^2 x_0 - F' \cos(\phi) ] + \cos(\omega t') [ \omega b' x_0 + F' \sin(\phi) ] = 0 \tag{65}
\]
Because there are both \(\cos(\omega t')\) terms and \(\sin(\omega t')\) terms in Equation 65, there is no way to make everything add up to zero for all times unless the coefficients of both \(\cos(\omega t')\) and \(\sin(\omega t')\) are separately equal to zero. This gives us the two equations
\[
F' \sin(\phi) = -\omega b' x_0 \tag{66}
\]
\[
F' \cos(\phi) = (\omega_0^2 - \omega^2) x_0 \tag{67}
\]
If we divide Equation 66 by Equation 67, the $F'$ and $x_0$ cancel and we get

$$\frac{\sin \phi}{\cos \phi} = \frac{-\omega b'}{\omega_0^2 - \omega^2}$$

$$\tan \phi = \frac{(b/m) \omega}{\omega^2 - \omega_0^2} \tag{59}$$

This is the result we stated earlier, namely Equation 59.

To solve for the amplitude $x_0$ of the oscillation, we can use Equation 66 to get

$$x_0 = \frac{F'}{\omega b'} \sin \phi \tag{68}$$

To find the $\sin \phi$ from the $\tan \phi$ which we already know, construct the right triangle shown in Figure (32). We have made the opposite and adjacent sides so that the ratio comes out as $\tan \phi$, and the hypotenuse is given by the Pythagorean theorem. Thus $\sin \phi$ is

$$\sin \phi = \frac{\omega b'}{\sqrt{(\omega^2 - \omega_0^2)^2 + \omega^2 b'^2}} \tag{69}$$

Substituting 69 into 68, and setting $b' = b/m$, $F' = F_d/m$, we get

$$x_0 = \frac{F_d/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + \omega^2 b'^2/m^2}} \tag{58}$$

which is our earlier Equation 58 for $x_0$.

**Transients**

In our qualitative discussion of forced harmonic motion, we saw that in addition to the driven oscillation $x_{\text{driven}} = x_0 \sin(\omega t + \phi)$ we have just studied, we could also have transient motion at the natural frequency $\omega_0$. In controlled experiments, you observe that any transient motion present initially finally dies out and you are eventually left with just the driven motion.

The transient motion $x_{\text{tr}}(t)$ is just damped harmonic motion that satisfies the equation of motion

$$\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = 0 \tag{48}$$

The question we wish to answer now is whether the forced harmonic motion equation

$$\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = F_d \sin \left(\omega t\right) \tag{54}$$

allows us to have both driven and transient motion at the same time. In other words, is a guess of the form

$$x = x_0 \sin \left(\omega t + \phi\right) + x_{\text{tr}} \tag{70}$$

where $x$ is the sum of the driven motion and an arbitrary amount of transient motion, is this sum also a solution of Equation 54?

If you substitute our new guess 70 into 54, the driven term satisfies the whole equation and the transient terms add up to zero because of Equation 48, thus we do get a solution. Transient motions are allowed by Newton’s second law.

![Figure 32](image_url)

*Figure 32*  
Triangle to go from $\tan \phi$ to $\sin \phi$. 

**Figure 32**  
Triangle to go from $\tan \phi$ to $\sin \phi$. 

---

[14-29]
In this appendix, we will use the computer to analyze the motion of the oscillating air cart shown in Figure (33). For this problem, the computer solution does not have the elegance of the calculus solution we have been discussing. The calculus approach gives us a single solution valid for all values of the experimental parameters. With the computer we have to alter the program and rerun it any time we want to change a parameter such as the mass of the cart, the spring constant, or the initial position or velocity. The calculus approach gives us a single formula valid for all values of the experimental parameters.

However, the advantage of using the computer is that we can easily modify the program to include new physical phenomena. For example, to add damping, all we have to do is change the command

\[
\text{LET } F = -kx
\]

to the command

\[
\text{LET } F = -kx - bV
\]

and rerun the program. To add damping to the calculus solution, we had to work with a differential equation (48) that was much more difficult to solve than the equation for undamped motion (11).

The computer opens up a number of possibilities for student project work. For example, in our discussion of the simple pendulum shown in Figure (21), we had to limit our analysis to small amplitude swings of the pendulum. For large amplitude swings, the restoring force became non-linear which led to a differential equation that is difficult to solve. As the amplitude increases, there is a lengthening of the period that is easy to measure but difficult to predict using calculus. However, with the computer, it is as easy to use the exact force \(Mg \sin \theta\) as it is to use the approximate linear force \(Mg \theta\). Thus, with the computer you can predict the lengthening of the period and compare your results with experiment.

In Appendix I, we made a considerable effort to predict the effects of adding a time dependent driving force to a harmonic oscillator. The work paid off in that we got Equations 58 and 59 which provide a general description of resonance phenomena. With the computer you do not get these elegant formulas, but it is much easier to add a time dependent force and see what happens. In effect, the computer solutions can be used as a laboratory to test the predictions of Equations 58 and 59. This provides an opportunity for a lot of project work.

**Figure 33**

Reproduction of Figure (10), showing an oscillating air cart. If the cart is displaced a distance \(x\) from equilibrium, there is a restoring force \(F = -kx\). The force is measured by adding weights as shown.
**English Program**

In Chapters 5 and 8 our usual approach for solving a new mechanics problem on the computer was to modify an old working program. But because the harmonic oscillator is an easy one dimensional problem, we will start over with a new program. Our general procedure has been to first write an English program that described the steps using familiar notation. Once we checked the steps to see that the program did what we wanted, we then translated the program into an actual computer language such as BASIC.

The English program for the oscillating air cart is shown in Figure 34. In the first section, we state the experimental constants, namely the mass M of the air cart and the spring restoring constant K. For this particular experiment, the cart has a mass M of 191 grams, and the spring constant K was 3947 dynes/cm. As indicated in Figure (33), the spring constant K was determined by tying a string to the mass, running the string over a pulley, and hanging weights on the other end. We got a linear force verses the distance curve like the one in Figure (9-4), and used the same method to find K.

In the next section of the program, we choose an explicit set of initial conditions. For this problem we start the cart from rest \( V_0 = 0 \) at a distance 10 cm to the right of the equilibrium position \( X_0 = 0 \). The cart is released at time \( T_0 = 0 \).

In the lab we observed that the period of oscillation was about 1.5 seconds. Thus a calculational time step \( dt = 0.01 \) seconds gives us about 150 points for one oscillation, enough points for a smooth plot.

The calculational loop is similar to the one in the projectile motion program of Figure (5-18), page 5-16, except that for one dimensional motion we do not need vectors, and the old command

\[ \text{LET } \vec{A} = \vec{g} \]

is replaced by

\[ \text{LET } F = -K \times X \]
\[ \text{LET } A = F / M \]

On the next page we repeat this English program and show its translation into the computer language BASIC.

**English Program**

! ---------- Experimental constants
\[
\text{LET } M = 191 \text{ grams (cart mass)} \\
\text{LET } K = 3947 \text{ dynes/cm (spring constant)}
\]

! ---------- Initial conditions
\[
\text{LET } X_0 = 10 \text{ cm} \\
\text{LET } V_0 = 0 \text{ (release from rest)} \\
\text{LET } T_0 = 0 \text{ (start clock)}
\]

! ---------- Computer Time Step
\[
\text{LET } dt = 0.01
\]

! ---------- Calculational loop
\[
\text{DO} \\
\text{LET } X_{\text{new}} = X_{\text{old}} + V_{\text{old}} \times dt \\
\text{LET } F = -K \times X \text{ (spring force)} \\
\text{LET } A = F / M \\
\text{LET } V_{\text{new}} = V_{\text{old}} + A_{\text{old}} \times dt \\
\text{LET } T_{\text{new}} = T_{\text{old}} + dt \\
\text{PLOT } X \text{ vs } T \\
\text{LOOP UNTIL } T > 15
\]

END

**Figure 34**
*English program for the motion of an oscillating cart on an air track.*
**English Program**

! --------- Experimental constants
LET M = 191 grams (cart mass)
LET K = 3947 dynes/cm (spring constant)

! --------- Initial conditions
LET X_0 = 10 cm
LET V_0 = 0 (release from rest)
LET T_0 = 0 (start clock)

! --------- Computer Time Step
LET dt = .01

! --------- Calculational loop
DO
  LET X_new = X_old + V_old*dt
  LET F = -K*X (spring force)
  LET A = F/M
  LET V_new = V_old + A_old*dt
  LET T_new = T_old + dt
  PLOT X vs T
LOOP UNTIL T > 15

END

*Figure 34 repeated*
*English program for the motion of an oscillating cart on an air track.*

**The BASIC Program**

Because no vectors are involved in the harmonic oscillator program, the translation into BASIC is almost automatic. Drop the subscripts “new” and “old”, fix up the PLOT statement, add the plotting window commands, and you have the result shown in Figure (35a). Select RUN and you get the plot of oscillating motion shown in Figure (35b).

**BASIC Program**

! --------- Plotting window
SET WINDOW -5,15,-15,15
PLOT LINES: -5,0; 15,0 ! T axis
PLOT LINES: 0,-15; 0,15 ! X axis

! --------- Experimental constants
LET M = 191
LET K = 3947

! --------- Initial conditions
LET X = 10
LET V = 0
LET T = 0

! --------- Computer Time Step
LET dt = .01

! --------- Calculational loop
DO
  LET X = X + V*dt
  LET F = -K*X
  LET A = F/M
  LET V = V + A*dt
  LET T = T + dt
  PLOT T,X;
LOOP UNTIL T > 15

END

*Figure 35a*
*BASIC program for the motion of an oscillating cart on an air track.*

*Figure 35b*
*Output of the BASIC program, showing the oscillation of the cart.*
The plot of Figure (35b) nicely shows the sinusoidal oscillation, but does not tell us the numerical value of the period of oscillation. To determine the period, we modified the program as shown in Figure (36a). The main change is to replace the PLOT statement by a PRINT statement. To reduce the output, we included MOD statement (as described in Exercise 5-5, page 5-9) so that only every tenth calculated point would be printed. From the output shown in Figure (36b), we see that the period is close to 1.4 seconds. A more accurate value of the period can be obtained by not using the MOD statement and printing every value as shown in Figure (36c). From this section of data we see that the period is closer to 1.39 seconds.

**Exercise 22**

Show that the frequency of oscillation seen in the computer output of Figure (36) is consistent with the calculus derived equation

\[ \omega = \sqrt{\frac{k}{M}} \]

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**Figure 36a**

Program for numerical output.

```plaintext
! -------- Print labels
PRINT 'T', 'X', 'U'

! -------- Experimental constants
LET M = 191
LET K = 3947

! -------- Initial conditions
LET X = 10
LET V = 0
LET T = 0

! -------- Computer Time Step
LET dt = .01
LET i = 0

! -------- Calculational loop
DO
  LET X = X + V*dt
  LET F = -K*X/M
  LET A = F/M
  LET U = U + A*dt
  LET T = T + dt
  LET i = i+1
  IF MOD(i,10) = 0 THEN PRINT T,X,U
LOOP UNTIL T > 3
END
```

**Figure 36b**

Numerical output.

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</tr>
<tr>
<td>1.42</td>
<td>9.09061</td>
<td>-7.60499</td>
</tr>
</tbody>
</table>

**Figure 36c**

Detailed numerical output. By printing every calculated numerical value, we can more accurately determine the period of oscillation.
Damped Harmonic Motion
In Figure (37a) we modified the projectile motion program of Figure (35a) to include damping. The only change, shown in boxes in Figure (37a) is to replace
\[
\text{LET } F = -K \times X
\]
by
\[
\text{LET } F = -K \times X - b \times V
\]
where we gave \( b \) the numerical value of 100 to get the result shown in Figure (35b).

Exercise 23
(This is more of an introduction to project work)
In our analysis of damped harmonic motion in Exercise 20, we predicted that the frequency for damped harmonic motion would be
\[
\omega = \sqrt{\frac{k}{M} - \frac{b^2}{4m^2}} \quad (51)
\]
In the special case that
\[
\frac{k}{M} = \frac{b^2}{4m^2} \quad b = \sqrt{4mk} \quad (51a)
\]
we get \( \omega = 0 \) which is the case of critical damping, where oscillations cease.

Run the damped harmonic oscillator program of Figure (35a) for values of \( b \) near \( \sqrt{4mk} \) and show that oscillations cease when you get to this critical value.

---

**Figure 37a**
BASIC program for the damped harmonic motion.

```
! Damped oscillation
! -------- Plotting window
SET WINDOW -5, 15, -15, 15
PLOT LINES: -5, 0; 15, 0 | T axis
PLOT LINES: 0, -15; 0, 15 | Y axis

! -------- Experimental constants
LET M = 191
LET K = 3947
LET b = 100

! -------- Initial conditions
LET X = 10
LET V = 0
LET T = 0

! -------- Computer Time Stop
LET dt = .01

! -------- Calculational loop
DO
  LET X = X + U * dt
  LET F = -K*X - b*U
  LET A = F / M
  LET U = U + A * dt
  LET T = T + dt
  PLOT T, X;
LOOP UNTIL T > 15
END
```

**Figure 37b**
Plot of damped harmonic motion.
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